University of Southern Queensland

Vector Solitons and their Interactions
Literature Review

A Thesis submitted by

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Abstract

Vector solitary waves, or solitons, of a specific type called helical solitons appear in various models of plasma physics and solids and play an important role in nonlinear wave dynamics. The helical solitons are less studied in contrast to other types (bell-shaped, table-top-shaped solitons, envelope solitons, and kinks). Preliminary investigations show that there are nontrivial interactions of plane solitons differently oriented in space and can be described by vector modified Korteweg–de Vries (vmKdV) equation. Two forms of the vmKdV equations are considered, integrable vmKdV and the non integrable vmKdV. The integrable vmKdV equation has exact analytical solution, non integralable vmKdV equation is analyzed using numerical methods. The structure and interactions of helical solitons with each other will be studied in this project by means of numerical modelling of the governing vmKdV equations. The results obtained will be compared with the analytical solutions of the integrable vmKdV equation.

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List of Abbreviations

$\frac{\partial u}{\partial t} \rightarrow u_t, \frac{\partial^2 u}{\partial t^2} \rightarrow u_{tt}$ First and second partial derivative of u with respect to t

$\frac{\partial u}{\partial x} \rightarrow u_x, \frac{\partial^2 u}{\partial x^2} \rightarrow u_{xx}$ First and second partial derivative of u with respect to x

$\frac{\partial^3 u}{\partial x^3} \rightarrow u_{xxx}$ Third partial derivative of u with respect to x

$\frac{\partial}{\partial t} \rightarrow \partial_t$ Partial derivative with respect to t

$\frac{\partial}{\partial x} \rightarrow \partial_x, \frac{\partial}{\partial x^2} \rightarrow \partial_{xx}, \frac{\partial}{\partial x^3} \rightarrow \partial_{xxx}$ First,second and third partial derivative with respect to x

∂ Partial derivative operator

FDM Finite Difference Method

ISM Inverse Scattering Method

KdV Korteweg–de Vries

KP Kadomtsev–Petviashvili

mKdV modified Korteweg–de Vries

NLS Nonlinear Schrödinger Equation

PDE Partial Differential Equation

SG Sine-Gordon

vmKdV vector modified Korteweg–de Vries

Dedicated to the memory of my mother, Mrs. Sumitra Srinivasa, whose unwavering belief in me was the foundation of my education and instilled a lifelong passion for mathematics

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Chapter 1

Introduction

Nonlinear wave theory has been widely adopted in the area of physics. The field of quantum mechanics demonstrated a relationship between waves and particles, further exploration in the area of classical physics led to the study of certain type of wave equations that tend to retain the shape and size indefinitely. Upon interaction, these waves preserve their shapes and speeds and propagate through the medium behaving like classical particles. Such types of waves that propagate through a medium preserving their characteristics are known as solitary waves or *solitons*. This literature review looks at the origins of the soliton theory, the governing mathematical equations and interactions among different soliton types.

The research project is aimed at analyzing a particular type of solitons viz., helical solitons that are less studied and analyzed compared to other known soliton types. Helical solitons arise, in particular, as a result of propagation of transverse non-linear waves in a chain of connected particles (the dynamical system model will be presented in the later sections of the document). The content of the review in chronological order is organized as follows:

Wave Theory: This section details the fundamental concepts of linear and nonlinear waves in dispersive media.

Soliton History: This section looks at the origins of solitons initially documented as observational science that later provided the framework for the formulation of Korteweg–de Vries (KdV) equation. The generalization of the KdV equation led to other equations and soliton types arising out of these equations are discussed.

Vector modified KdV Equation: This section looks at the modification to the KdV equation by introducing a vector component and the resulting behavior of waves that will lead to formation of helical solitons. The structure and interactions of helical solitons with each other will be studied in this research project. The results from numerical modelling will be compared with analytical solutions for the validation.

Chapter 2

Wave Theory

A wave can be defined as a set of oscillations propagating through a medium that carry a signal or energy from one part of the medium to another at a particular velocity. The propagation of a wave through a medium is associated with changes in some of the properties, but remains recognizable [5]. The time evolution phenomenon of a wave is mathematically modeled using *linear and nonlinear partial differential equations (PDEs)*, with wave profile represented by $u(x,t)$ where x is spatial coordinate and t is the time coordinate. Some examples of physical wave types and their occurrences include [26]:

- **Acoustic waves** - audible sound, medical applications of ultrasound.
- **Electromagnetic waves** - electricity in various forms, radio waves, light waves in optical fibers.
- **Seismic waves** - resulting from earthquakes in the form of P waves and S waves.
- **Water waves** - Capillary (short waves of a typical wavelength $\lambda \sim 2.5$ cm) and gravity waves (which exist due to the restoring force from the Earth' gravity field) with wave lengths greater than 2.5 cm. Gravity waves in turn can be classified as deep water waves and shallow water waves depending on the relationship between the wavelength and water depth.
- **Gravitational waves** - Waves that are invisible and occur as fast ripples in space traveling at the speed of light 1.86×10^5 miles/sec and are result of objects moving at high speeds

some examples of events where gravitational waves occur include two big stars orbiting each other and a star exploding asymmetrically (supernova). The gravitational waves are detected through use of sensitive instrument called LIGO (Laser Interferometer Gravitational-Wave Observatory).

A well-understood form of wave motion consists of a pattern of crests and troughs representing a sinusoidal wave, and this is commonly referred to as linear wave having the following characteristic properties:

- The velocity and shape are independent of the amplitude.
- The sum of two linear waves will result in another linear wave.
- Linear waves have a small amplitude.

In contrast, a nonlinear wave is a large-amplitude wave and there will be distortion of the shape of the wave as it propagates through a medium [27]. A representation of wave propagating through a medium is shown in Figure 2.1.

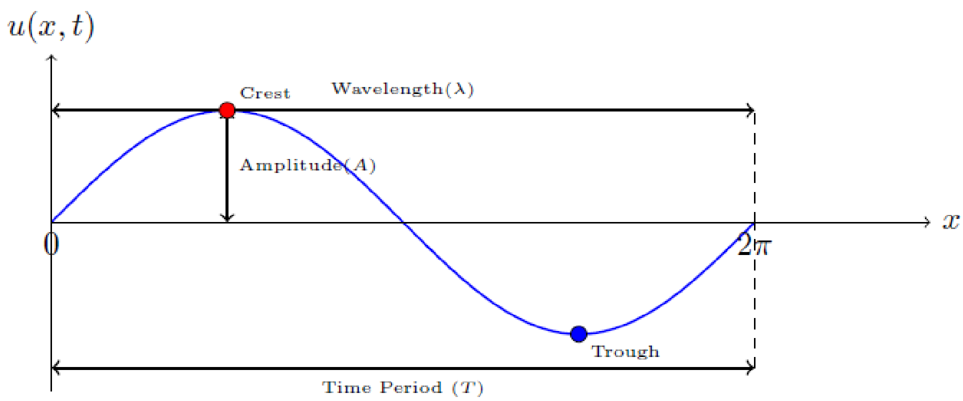


Figure 2.1: Wave form

The main attributes of a wave propagating through a medium include:

- *Wavelength* (λ): the distance between adjacent crests
- *Wave number* (κ): the ratio of one complete wave cycle to wavelength ($\kappa = \frac{2\pi}{\lambda}$)
- *Time period* (T): the time taken to complete one oscillation

- *Frequency* ($f = \frac{1}{T}$): number of oscillations completed per unit time
- *Angular frequency* ($\omega = \frac{2\pi}{T}$): the angular displacement of element of wave per unit time

Waves are categorized into two main types viz., hyperbolic waves and dispersive waves. A prototype representation of hyperbolic waves is considered to be a one-dimensional wave equation that is linear second-order partial differential equation and satisfies the principle of superposition [1, 5].

$$u_{tt} = c^2 u_{xx} \tag{2.1}$$

$u(x,t)$ is the amplitude of the wave and c is a positive constant. A general solution of (2.1) is the well known as *d'Alembert's* solution of the form

$$u(x,t) = f(x - ct) + g(x + ct). \tag{2.2}$$

where f and g are arbitrary functions. The solution consists of two non interacting waves, $f(x-ct)$ refers to right moving wave and $g(x+ct)$ represents a left moving wave at a constant speed c. The arbitrary functions f and g can be determined by initial conditions $u(x,0)$ and $u_t|_{t=0}$ [8, 4]. The one-dimensional wave equation (2.1) can be rearranged to the following form:

$$(\partial_t - c\,\partial_x)\,(\partial_t + c\,\partial_x)\,u = 0 \tag{2.3}$$

From (2.3) we have the following first order equations

$$u_t - c\,u_x = 0 \tag{2.4}$$

$$u_t + c\,u_x = 0 \tag{2.5}$$

The equations (2.4) and (2.5) can be solved using the method of characteristics [4] resulting in general solutions of the form $u(x,t) = g(x + ct)$ and $u(x,t) = f(x - ct)$ respectively where g and f are arbitrary functions determined by initial conditions. Thus we see that for the linear second order wave equation (2.1) and the first order equations (2.4) and (2.5) the solution consists of a wave train moving either in the left or right direction at constant speed c and the waves do not interact with each other. Travelling waves applicable to non-linear equations are waves in which the medium moves along the direction of propagation of wave. There are three type of travelling waves depending on the shape of the wave [8, 16] shown in Figure 2.2.

- Wave trains that are spatially periodic, i.e., $f(x+p) = f(x)$ for $p > 0$.
- A wave pulse that is asymptotically constant, i.e., $\lim_{x \to \pm\infty} u(x)$ and $A^+ = A^-$ where A is a constant.
- A wave front or wave back will be asymptotically constant, but $A^+ \neq A^-$.

Figure 2.2: Shapes of travelling wave(a), pulse (b), spatially-periodic wave (c), anti-kink and kink wave (d) (from [16]).

2.1 Phase and Group Velocities

As the wave propagates through the medium, the crests and troughs of the wave components are enclosed within an envelope [1] as shown in Figure 2.3.

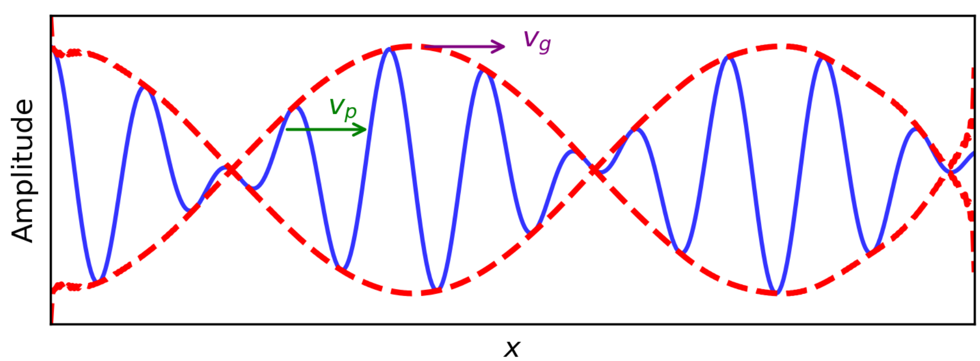


Figure 2.3: Wave envelope

Two different velocity components are identified for propagating wave viz., *phase velocity (v_p) and group velocity (v_g)* [1, 3]. The *phase velocity (v_p)* refers to the velocity of individual wave component and is expressed in terms of angular frequency (ω) and wave number (κ) by the relation:

$$v_p = \omega/\kappa \tag{2.6}$$

The *group velocity (v_g)* refers to the velocity of the wave packets within the envelop and is given by:

$$v_g = d\omega/dk \tag{2.7}$$

From (2.6), we obtain $\omega = \kappa v_p$. Differentiating with respect to κ, we obtain the following:

$$d\omega/dk = v_p + \kappa\, dv_p/dk \tag{2.8}$$

Using (2.7) we can establish the following relation between group velocity (v_g) and phase velocity (v_p)

$$v_g = v_p + \kappa\, dv_p/dk \tag{2.9}$$

If group velocity (v_g) and phase velocity (v_p) are equal then it results in a non-dispersive wave. However if the velocities are different then the waves spread out as they propagate through the medium, such type of waves are referred to as dispersive waves. In the next section we consider equations that exhibit these wave properties.

2.1.1 Dispersionless Waves

A wave is said to be *dispersive* if the wave components propagate with different phase velocities, resulting in localized wave packet to spread as it propagates through a medium. As a result, the amplitude of wave packet decreases, while the total energy is conserved but redistributed over a larger region of space. To examine dispersive nature of wave, a plane wave of the form $u(x,t) = e^{j(kx-\omega t)}$, where k is the wave number and ω is the frequency is considered as a good *ansatz* to the different wave equations. A *dispersion relation* describes the relationship between the frequency ω and the wave number k i.e, $\omega = \omega(k)$ and represents a relation between the energy of the system and momentum.The necessary conditions for dispersion are $\omega(k)$ must be real and $\omega''(k) \neq 0$ [4, 7, 3]. For the one-dimensional wave equation (2.1) the dispersion relation can be obtained by calculating u_{tt} and u_{xx} from the plane wave solution.

$$u_t = -j\omega e^{j(kx-\omega t)}; \quad u_{tt} = -\omega^2 e^{j(kx-\omega t)} \tag{2.10}$$

$$u_x = jke^{j(kx-\omega t)}; \quad u_{xx} = -k^2 e^{j(kx-\omega t)} \tag{2.11}$$

Substituting for u_{xx} and u_{tt} in (2.1) we obtain the linear relation $(\omega^2 = c^2k^2) \Rightarrow \omega = \pm ck$ shown in Figure 2.4. The *phase velocity* $(v_p = \omega/k = \pm c)$ and *group velocity* $(v_g = d\omega/dk = \pm c)$ are equal. Substituting $\omega = \pm ck$, we obtain $u(x,t) = e^{jk(x \pm ct)}$. The solution has two components, a right moving $(x - ct)$ or left moving $(x + ct)$ wave with a constant speed c and therefore the linear one-dimensional wave equation (2.1) is dispersionless i.e, the waves do not spread out as they propagate.

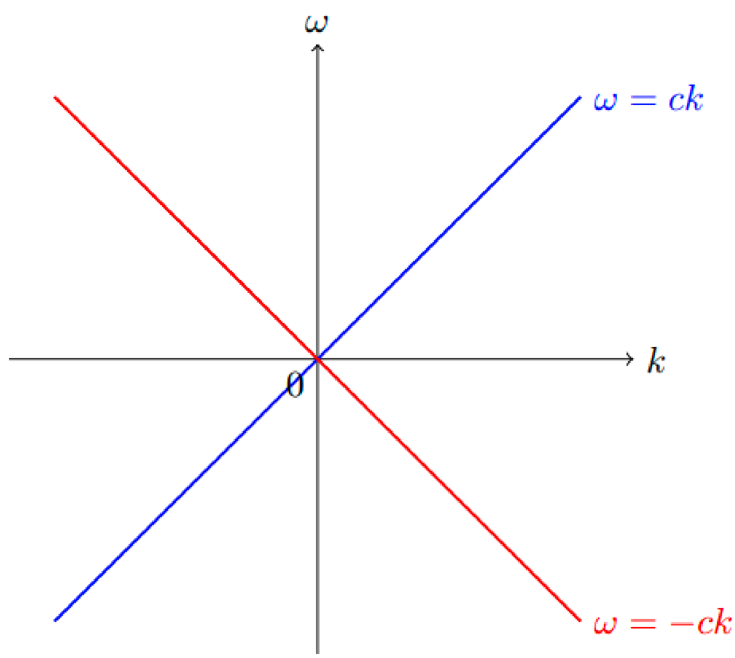


Figure 2.4: Plot of ω and k

2.1.2 Dispersive and Dissipative Waves

To examine the *dispersion* and *dissipation* properties of wave propagation we consider the following linear partial different equations [1].

$$u_t + u_x + u_{xxx} = 0 \tag{2.12}$$

$$u_t + u_x - u_{xx} = 0 \tag{2.13}$$

Using *ansatz* $u(x,t) = e^{j(kx-\omega t)}$ we obtain the following partial derivatives:

$$u_t \Rightarrow -i\omega e^{j(kx-\omega t)}, \quad u_x \Rightarrow ike^{j(kx-\omega t)}, \quad u_{xx} \Rightarrow -k^2 e^{j(kx-\omega t)}, \quad u_{xxx} \Rightarrow -ik^3 e^{j(kx-\omega t)}$$

Substituting the above in (2.12)

$$-i\omega e^{j(kx-\omega t)} + ike^{j(kx-\omega t)} - ik^3 e^{j(kx-\omega t)} = 0 \Rightarrow \omega(k) = k - k^3$$

Substituting for $\omega(k)$ in *ansatz* for $u(x,t)$ we obtain the following:

$$u(x,t) = e^{i(kx-(k-k^3)t)} = e^{ik(x-(1-k^2)t)}$$

The phase velocity $v_p = \omega(k)/k = 1 - k^2$ and group velocity $v_g = d\omega/dk = 1 - 3k^2$. Since $v_p \neq v_g$ and $\frac{d^2\omega}{dk^2} \neq 0$, this implies that wave components travel at different velocities and (2.12) represents a dispersive wave and the general solution is given by [1]:

$$u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega(k)t)} dk, \qquad \omega(k) = k - k^3 \tag{2.14}$$

$A(k)$ represents the fourier transform of initial condition $u(x,0)$. A representation of a dispersive wave for the initial condition $u(x,0) = e^{-x^2/4}$ is shown in Figure 2.5.

In (2.13) for even derivative of u with respect to x, the equation changes to the following after substituting the partial derivatives:

$$-i\omega e^{j(kx-\omega t)} + ike^{j(kx-\omega t)} + k^3 e^{j(kx-\omega t)} = 0 \Rightarrow \omega(k) = k - ik^2$$

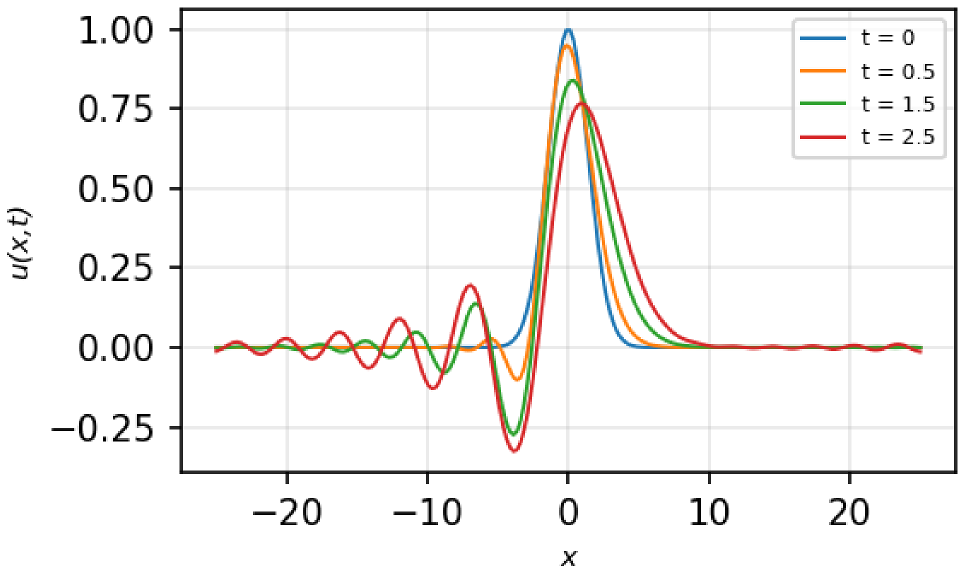


Figure 2.5: Dispersive wave

Substituting for $\omega(k)$ in *ansatz* for $u(x,t)$ we obtain the following:

$$u(x,t) = e^{i(kx-(k-ik^2))t} = e^{-k^2t+i(k-t)}$$

The phase velocity $v_p = \omega(k)/k = 1 - ik$ considering only the real value for $\omega(k)$ we can establish that phase velocity v_p has a constant value of 1 and therefore the wave propagates a speed of unity for all values of wave number k. In addition, as $t \to \infty$, there is an exponential decay due to e^{-k^2t}, and such types of waves are referred to as *dissipative waves* is shown in Figure 2.6. The observation from (2.12) and (2.13) is that for odd powers in spatial derivatives there will be dispersion and for even powers in spatial derivatives there will be dissipation.

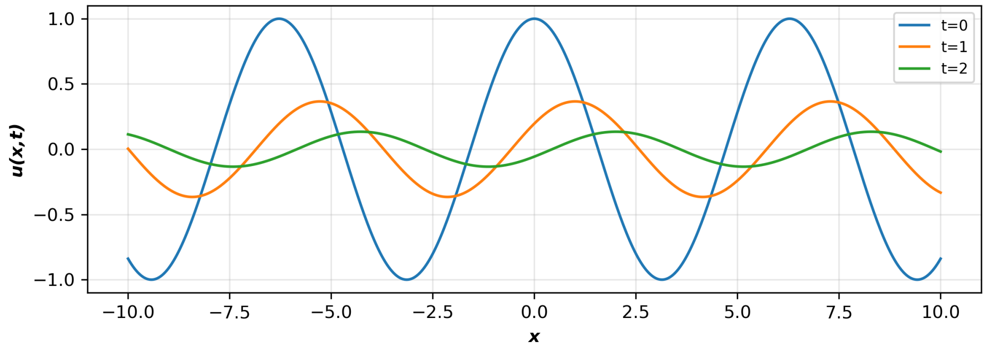


Figure 2.6: Exponential decay (dissipation)

2.1.3 Nonlinear Dispersionless Waves

To examine non-linear effects on wave motion we consider equations containing the non-linear term uu_x. The following equation is taken from [1].

$$u_t + uu_x + u_x = 0 \quad \Rightarrow u_t + (1+u)u_x = 0 \tag{2.15}$$

The above equation is a quasi-linear partial differential equation can be solved using the method of characteristics [3], the general solution to (2.15) can be represented as $u(x,t) = f(x-(1+u)t)$ where f is an arbitrary function determined by the given initial condition.The general solution $u(x,t)$ will be single-valued for a finite time and unique, and beyond a certain time it will result in a non-unique solutions (multivalued), this results in a wave displaying *discontinuity* and and are referred to as a *"shock waves"* shown in Figure 2.7.

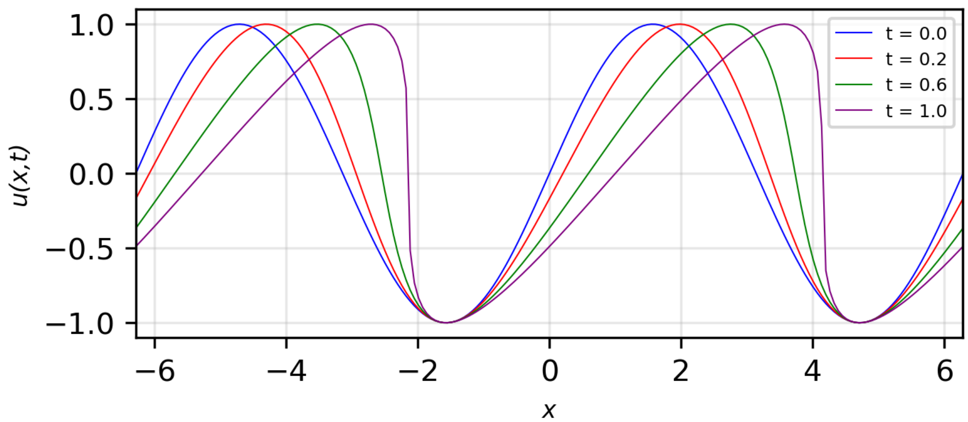


Figure 2.7: Wave discontinuity $u(x, 0) = sin(x)$

In the next chapter we will look at the history of Solitons and provide overview of Korteweg–de Vries (KdV) equation.

Chapter 3

Soliton History

Solitons or solitary waves are the solutions to certain class of non-linear partial differential equations. The localized solutions of integrable equations are referred to as "*solitons*", while localized solutions of non-integrable equations are called *"solitary waves"* [27]. Nonetheless, in physical application, such difference is ignored and both these terms are considered as synonyms. A solitary wave was first observed by John Scott Russell in the Edinburgh–Glasgow canal in August 1834 which he described as *"a great wave of translation"*. He left the following description of the observed phenomenon [2]:

"I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prowl of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the winding's of the channel. Such, in the month of August 1834, was my first chance interview with that singular

and beautiful phenomenon which I have called the Wave of Translation."

John Scott Russell further carried out a series of experiments to simulate solitary waves in a water tank.

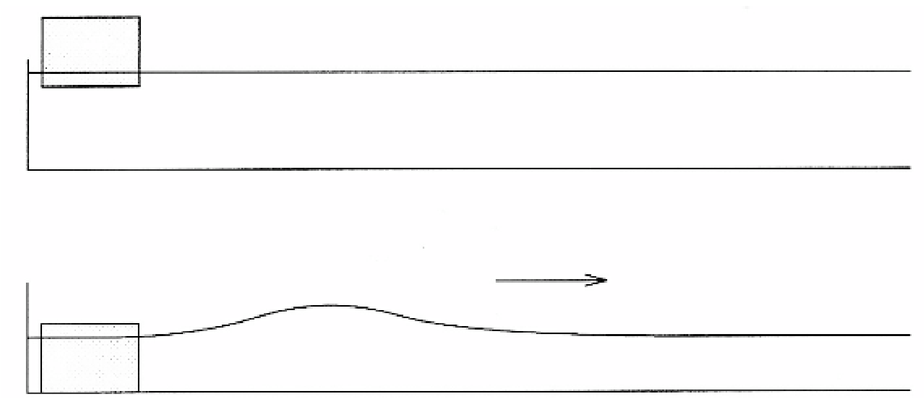


Figure 3.1: Russell's experiment to generate a solitary wave. (From [9]).

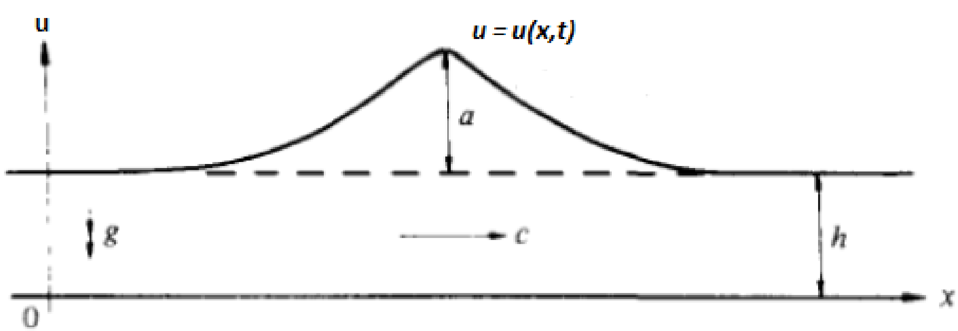


Figure 3.2: Solitary wave parameters (From [1]).

On the basis of these experiments, Russel presented a Report where he summarized his findings [1, 14, 27]:

- *The waves generated have constant speed and propagate over long distances preserving shape and speed upon collision.*
- *The speed of the wave is directly proportional to amplitude and hence larger amplitude waves travel faster.*

- *The speed of the wave can be described by the formula c^2=$g(h+a)$ and solitary waves were referred to as gravity waves.*

- *The waves are localised and either decays or approaches constant at infinity.*

- *There can be only solitary elevations; solitary cavities (depressions) never occur.*

Then, Boussinesq in 1871 and Rayleigh in 1876 gave the first theoretical description of solitary waves by considering that the length of solitary waves is much greater than the depth of the water with the following expression for perturbation of a water surface $u(x,t)$ [3, 1]:

$$u(x,t) = a\,\mathrm{sech}^2\,(\beta - ct), \quad \beta^2 = \frac{3a}{h^3}. \tag{3.1}$$

This solution is valid only if soliton amplitude a is much less than the water depth h, i.e., $a/h \ll 1$. The mathematical equation possessing solitary solutions was derived by Korteweg and de Vries in 1895 and is known as the Korteweg–de Vries (KdV) equation. In the dimensionless variables it has the form [8, 3]:

$$u_t + 6u_x + u_{xxx} = 0 \tag{3.2}$$

where $u(x,t)$ is the dimensionless water surface perturbation. The similar KdV equation was later derived to describe weakly dispersive and weakly nonlinear wave processes in various physical systems such as plasma physics, oceanography, solid state physics, and nonlinear optics [3].

Zabusky and Kruskal in 1965 numerically studied the formation of solitary wave within the KdV equation from the sinusoidal initial perturbation as shown in Figure 3.3. They described their findings in three steps [6]:

1. Initially the terms $u_t + uu_x$ result in the steepening of the function $u(x,t)$ in regions of negative slope resulting in discontinuity.

2. In the second phase, the term u_{xxx} becomes dominant and prevent formation of discontinuity. Instead of the wave overturning, oscillations of a small wave length are formed that attain steady state amplitudes.

3. Then, *"Solitary waves"* are formed and move with the speeds linearly proportional to their amplitudes.

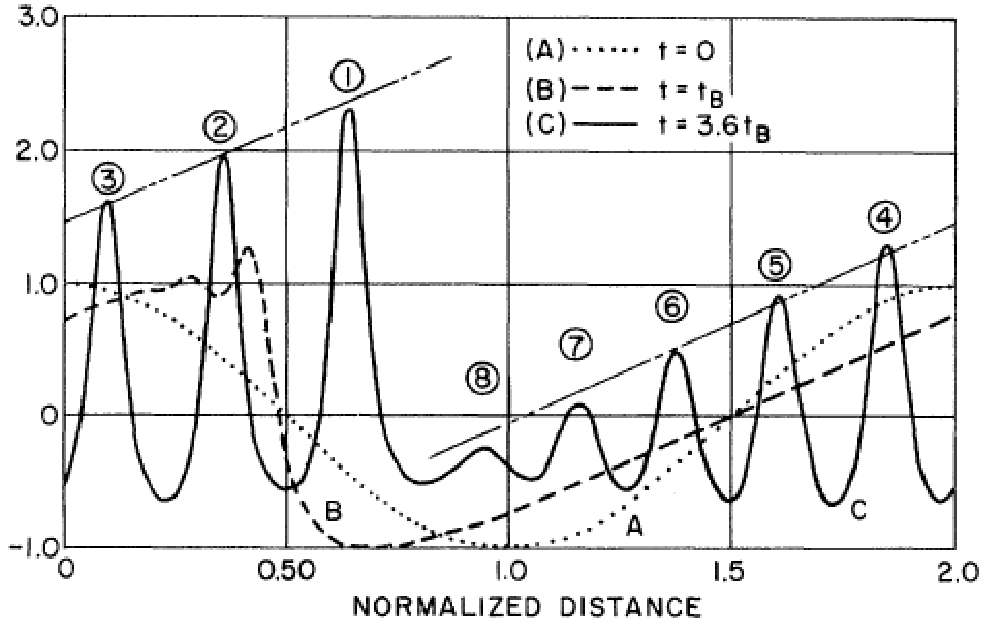


Figure 3.3: Time evolution of $u(x,t)$ from the sinusoidal initial condition – see the dotted line A. (From [6]).

The *"solitary wave pulses"* retain their identities and resemble a particle-like behavior; hence, they are called *"solitons"*. As follows from Eq. (3.2), the KdV equation contains three terms; the first one, u_t represents the time evolution, the second one, uu_x represents the nonlinear term, and the third one, u_{xxx} is the dispersive term. The behavior of a wave profile can be analyzed by considering the non-linear and dispersive terms separately in conjunction with the time evolution. The balance between the nonlinearity and dispersion is responsible for the existence of solitons.

There are only certain classes of nonlinear partial differential equations that admit soliton solutions; they are referred to as *"completely integrable systems"*. The conditions under which solitary waves can be obtained is summarized in Figure 3.4.

In the next section the properties of KdV equation (3.2) are detailed along with "*one-soliton*" and "*two-soliton*" solutions.

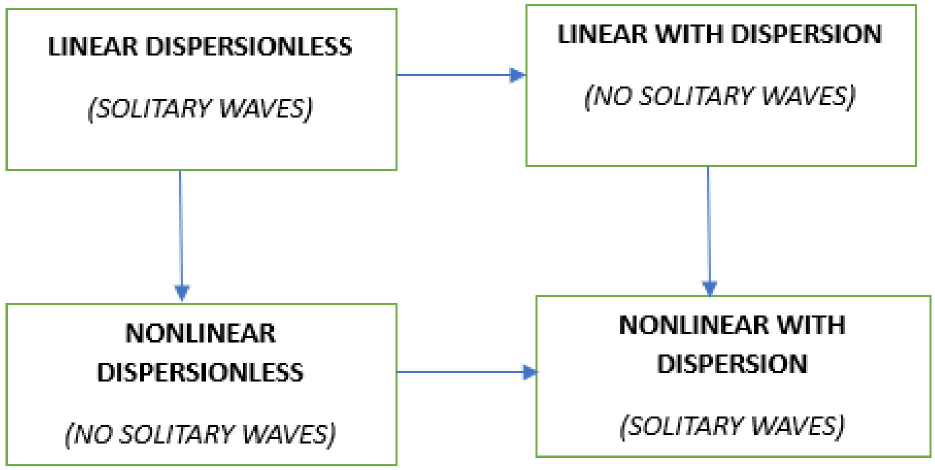


Figure 3.4: Conditions which admit existence of solitary waves (from [10]).

3.1 Soliton solutions of the KdV equation

The KdV equation (3.2) is completely integrable; its particular exact solutions can be found by considering a travelling wave solution of the form $u(x,t) = f(x - ct)$. For the one-soliton the solution is [8]:

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right), \tag{3.3}$$

where c is the wave speed and x_0 is the phase shift. A temporal development of $u(x,t)$ with $c = 4$ and $x_0 = 0$ demonstrates a wave travelling in the form of a bell-shaped pulse having a fixed amplitude and propagating with a constant speed. The wave pulse is localized so that $u(x,t) \to 0$ when $x \to \pm\infty$ [14] shown in Figure 3.5. The properties of KdV soliton can be summarized as follows [27]:

- *The amplitude and velocity are directly proportional and therefore large amplitude wave pulse travels faster.*
- *The wave pulse is stationary; its shape does not change in the process of propagation. In the co-moving coordinate frame it looks like an immovable hill.*
- *Solitons are symmetrical at the point where they have maximums.*

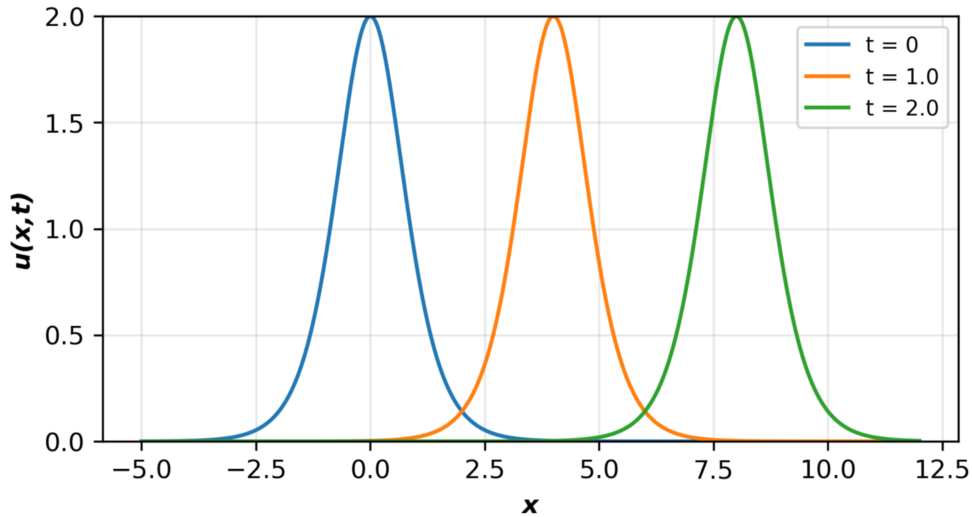
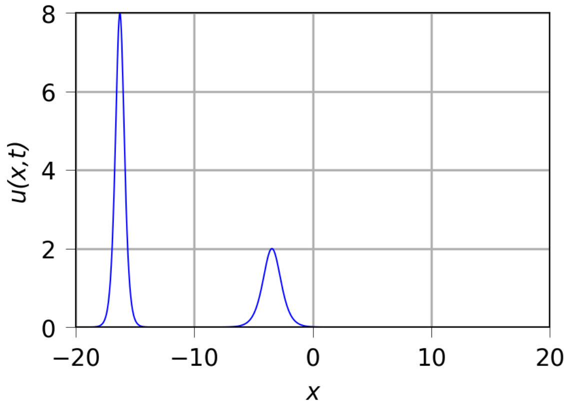


Figure 3.5: Time evolution of soliton moving from left to right.

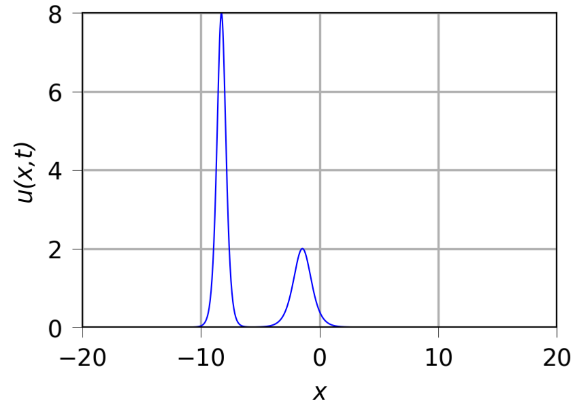
The interaction of two solitons moving with different speeds is elastic and resembles a particle-like behavior. The principle of superposition is not applicable to the KdV equation because this equation is nonlinear. However, there are several methods allowing us to describe an interaction of two-solitons. In particular, the Inverse Scattering Method (ISM), Hirota transform method, and some others [2, 1]. The two-soliton solution is given by the following expression [14, 7]:

$$u(x,t) = \frac{12[3 + 4cosh(2x - 8t) + cosh(4x - 64t)]}{[3cosh(x - 28t) + cosh(3x - 36t)]^2} \tag{3.4}$$

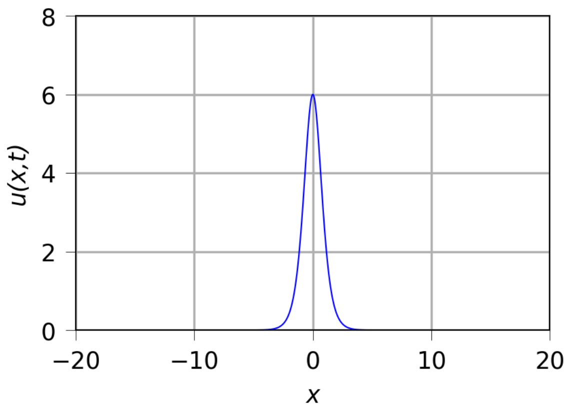
A 2-soliton interaction plot of (3.4) at different times t is shown in Figure 3.6.



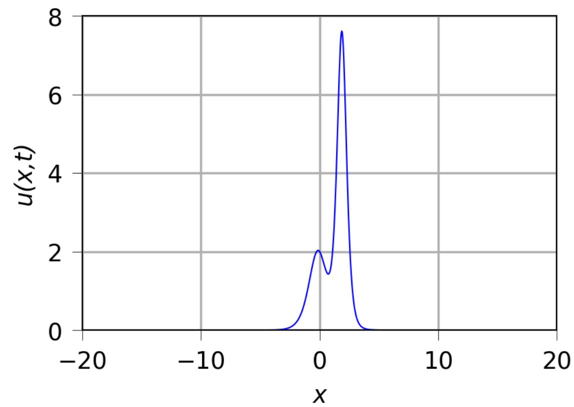
(a) $t = -1$



(b) $t = -0.5$



(c) $t = 0$



(d) $t = 0.1$

Figure 3.6: Two-soliton solution at different times.

The characteristics of nonlinear interaction of two KdV solitons depend on their speed (or amplitude) ratio c_1/c_2. The following types of interaction are possible [17, 18]:

- If $c_1/c_2 < 3$ the solitary waves experience a bounce and exchange speeds and shapes.
- If $c_1/c_2 > 3$ the solitary wave peaks merge first together, and then split apart restoring their initial shapes and speeds.

In both cases, solitons experience spatial and temporal phase shifts. The bigger soliton shifts ahead, and a smaller soliton shifts back. In the next section, different soliton types will be described that arise in different nonlinear PDE's.

Chapter 4

Soliton types

There are two broad classifications of solitons viz., topological and non-topological solitons. Topological solitons are localized particle-like objects and superimpose nonlinearly retaining identity after interaction and are therefore extremely stable. The velocity of the propagating wave does not depend on the amplitude with boundary conditions at ∞ different from that of the physical vacuum state. Topological solitons can be further divided into two subgroups taking into account the time-dependent or permanent nature of the profile. A non-topological soliton will have the boundary conditions at ∞ similar to that of the physical vacuum state. The physcial vacuum state refers to the trivial background solution in which $u(x,t) \to 0$ as $x \to \pm\infty$ [14, 15, 21]. Some of the different types of solitons arising out of characteristics of nonlinear partial differential equations are detailed below.

4.1 Kink, antikink, and breathers

Kink and antikink solitons are topological solitons. They represent one-dimensional solitary waves, and the solution value changes due to a transition from one state to another. A breather refers to nonlinear localized excitations wherein energy accumulates in a bounded and oscillatory manner. Once maximum amplitude is attained, the breather decays symmetrically on both sides of the point of maximum amplitude. These types of soliton solutions are obtained from the Sine-Gordon equation and appear in nonlinear optics, mechanical transmission lines, Josephson junctions, and Bloch wall motion of magnetic crystals. The Sine-Gordon equation is of the form [14, 19]:

$$u_{tt} - u_{xx} + sin(u) = 0, \quad -\infty < x < \infty, \quad 0 < t < \infty \tag{4.1}$$

The *kink and antikink* soliton solutions of (4.1) are given by [19]:

$$u(x,t) = \begin{cases} 4\tan^{-1}\left(e^{+\gamma(x-ct)}\right) & \text{(4.2a)} \\ 4\tan^{-1}\left(e^{-\gamma(x-ct)}\right) & \text{(4.2b)} \end{cases}$$

where $\gamma = \frac{1}{\sqrt{1-c^2}}$ is the Lorenz contraction factor. Antikink soliton (4.2a) represents a 2π pulse, and kink soliton (4.2b) represents a -2π pulse shown in Figure 4.1. The derivative of both kink and antikink has the shape of soliton [10] shown in Figures 4.2 and 4.3.

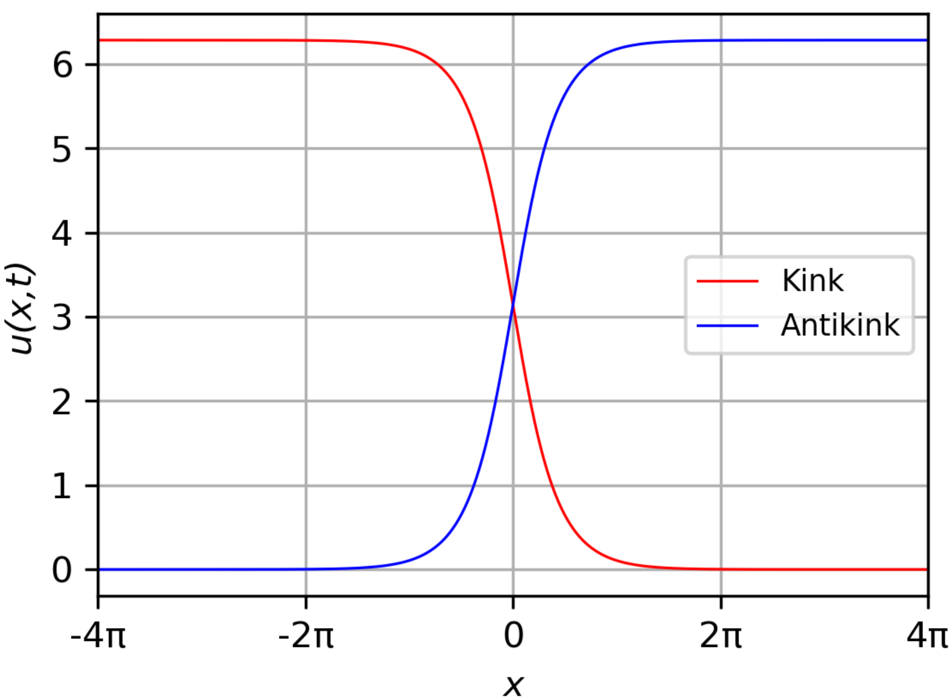


Figure 4.1: Kink and antikink soliton.

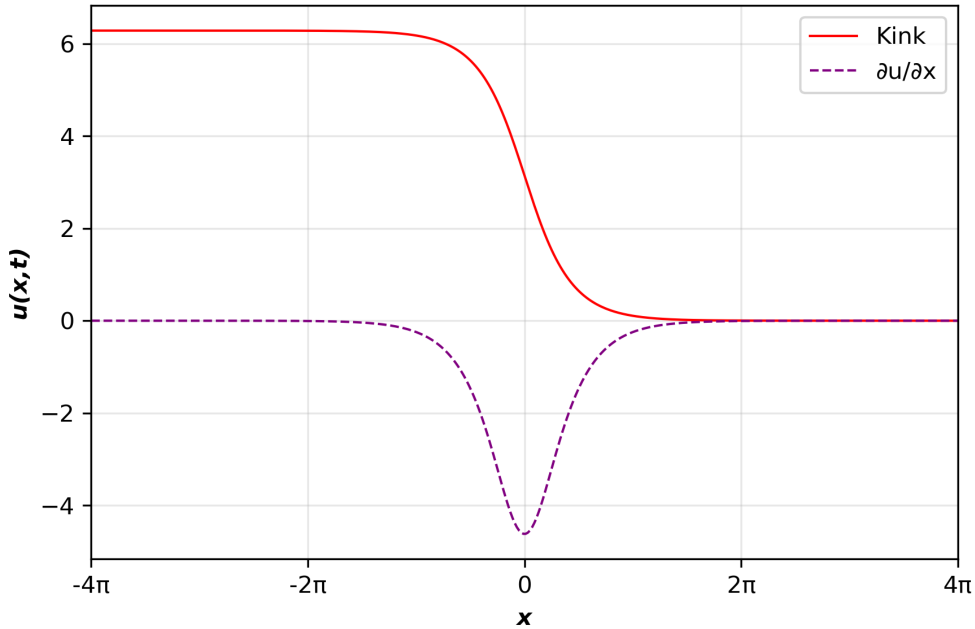


Figure 4.2: Derivative of kink

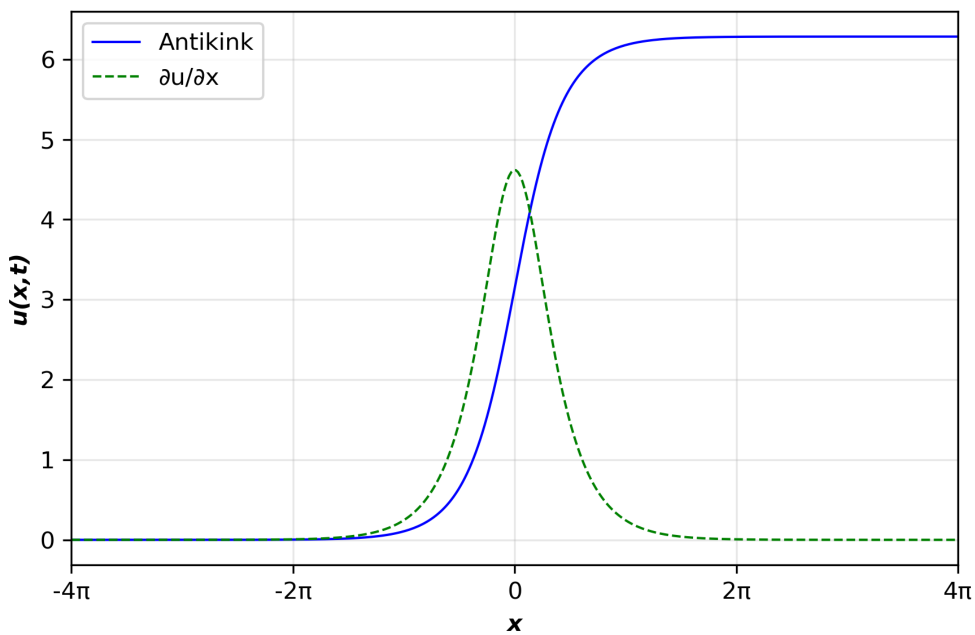


Figure 4.3: Derivative of antikink

Other solutions of Sine-Gordon equation (4.1) demonstrate soliton interactions including kink-kink collision and symmetric kink-antikink collision. For *the kink-antikink* solution, if the velocity parameter c is considered imaginary, we obtain a bound state solution known as stationary breathers, which is spatially localized and oscillates in time with frequency ω is shown in Figure 4.4. The breather solution is of the form [19, 32].

$$u(x,t) = 4 \arctan \left[\frac{\sqrt{1-\omega^2}}{\omega} \sin(\omega t) \operatorname{sech}\left(\sqrt{1-\omega^2} x\right) \right] \tag{4.3}$$

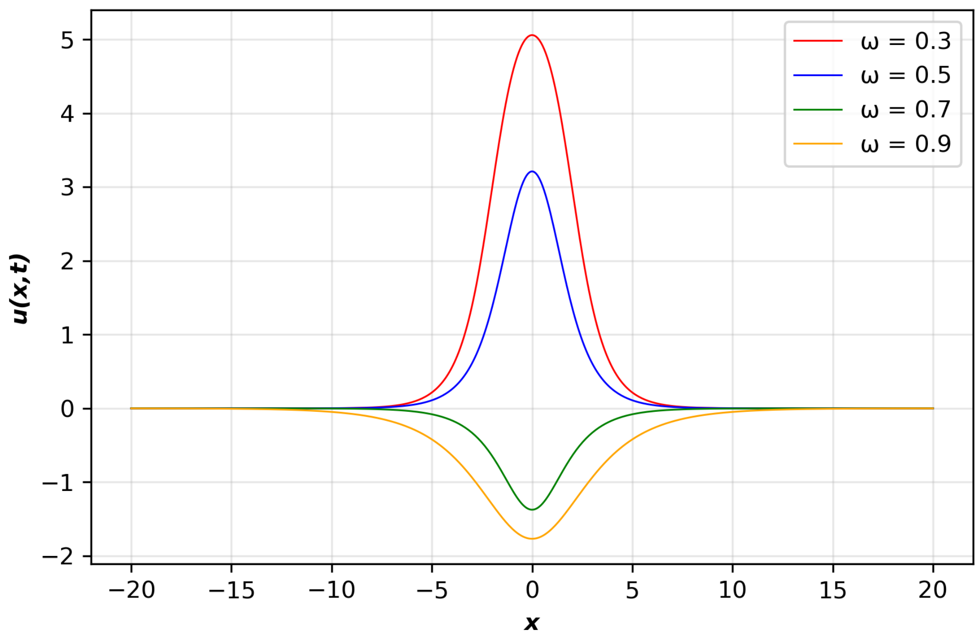


Figure 4.4: Breather oscillations at various frequencies

4.2 Envelop Solitons

Envelop solitons are solitary wave solutions that occur in a dispersive nonlinear medium. They are further classified into "bright" solitons that have localized shape and decay at ∞, and "dark" solitons having a constant amplitude at ∞ [14, 3]. The envelop soliton is presented by nonlinear Schrödinger equation (NLS) is given by:

$$iu_t + u_{xx} + \gamma |u|^2 u = 0, \quad \gamma \text{ is a constant, } u(x,t) \text{ is complex} \tag{4.4}$$

For an exact solution of (4.4) in the form $u(x,t) = \sqrt{2}\cos(2x - 3t)\,\mathrm{sech}(x - 4t)$ [14], the envelop soliton is shown in Figure 4.5.

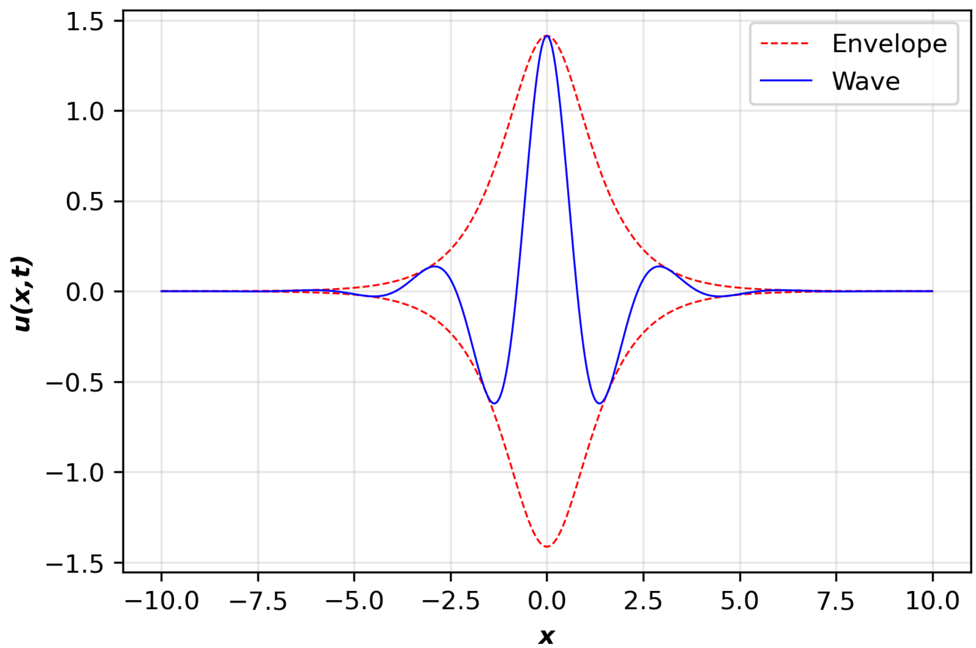


Figure 4.5: Envelop soliton of NLS equation

4.3 Lump Solitons

A generalized form of KdV equation (3.2) was formulated for function in two space dimensions and one time dimension ($u \Rightarrow u(x,y,t)$) in 1970 by Kadomstev-Petviashvili to describe weakly dispersive waves and this equation is the well known Kadomstev-Petviashvili (KP) equation of the form [20]:

$$\partial_x (u_t + 6uu_x + u_{xxx}) + 3\sigma u_{yy} = 0 \tag{4.5}$$

Two forms of (4.5) are defined viz., KP1 for $\sigma = 1$ and KP2 for $\sigma = -1$. The KP1 equation is completely integrable and has soliton solutions called *"lumps"* that are considered stable with respect to interactions, whereas the KP2 equation is non integrable however lump solutions for KP2 have been constructed numerically. Both these forms of equations have relevance in surface and internal water waves, optical waves, elastic waves in thin plates, etc. An example of lump solution is shown in Figure 4.6.

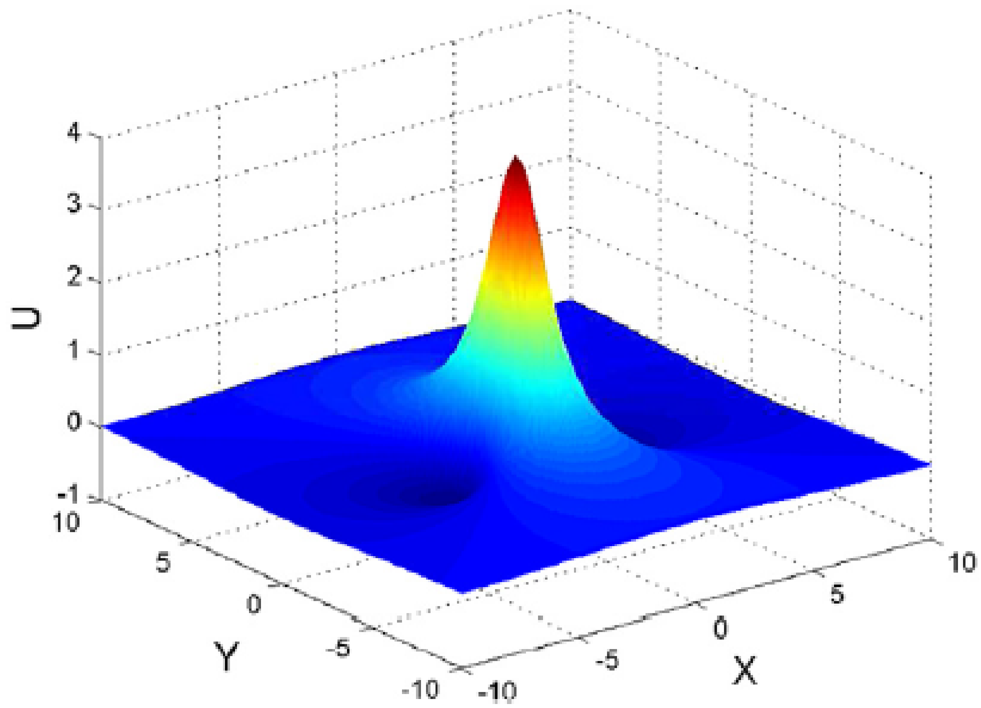


Figure 4.6: Lump soliton (From [20]).

In addition there are other types of solitons viz., "fat" and table-top solitons which are particular solutions of Gardner equation, Kawahara solitons with oscillating tails that are solutions of generalised KdV equation containing third and fifth order dispersion terms [30] shown in Figure 4.7. In the next chapter we consider helical solitons that change their polarisation during the course of propagation.

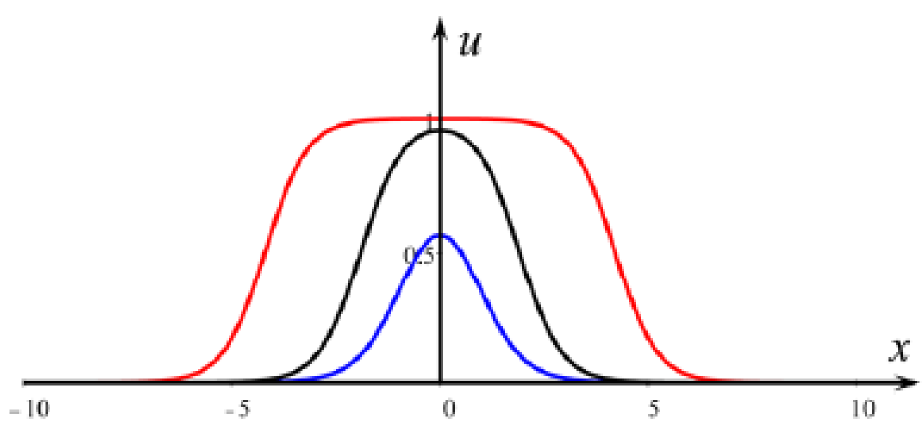


Figure 4.7: Kdv (blue), fat (black) and table-top (red) solitons (From [30]).

Chapter 5

Helical Solitons

5.1 Introduction

The type of solitons that change the orientation in space during the course of propagation are referred to as a helical solitons. Such types of waves are known to exists in electrodynamics, plasma physics, magnetohydrodynamics. Several papers have been published citing the occurence of helical solitons with earliest example being for perturbations along a thin vortex filament in a fluid by Hasimoto (1972) [22]. Gorshkov et al. [24] identified helical soliton solutions for circularly polarized electromagnetic waves in nonlinear isotropic dispersive media. The solutions obtained represented localized wave pulses and the duration of these wave pulses was comparable to rotation period of field vector.

Karney et al. [23] investigated vector solitons for lower hybrid waves in plasma looking at the close relevance between the complex modified KdV equation (CMKDV) and the modified KdV equation. Two types of solitary waves were observed, a constant wave pulse and an envelope solitary wave. An equation describing propagation of hybrid waves in homogeneous plasma was derived, represented in the following form:

$$\mathbf{u}_t + \left(|\mathbf{u}|^2\mathbf{u}\right)_x + \mathbf{u}_{xxx} = 0 \tag{5.1}$$

The two component vector $\mathbf{u}(x,t) = u_1(x,t) + i\,u_2(x,t)$ or in the polar form $\mathbf{u}(x,t) = r(x,t)e^{i\Theta(x,t)}$ has the norm $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$. Equation (5.1) is also referred to as the vector modified KdV (vmKdV) equation is non integrable and is analyzed numerically. The equation was also derived for a dynamical system consisting of a chain of connected particles by an elastic string shown in Figure 5.1. The details of derivation are included in [12, 25]. The displacement occurs in two perpendicular directions viz., y and z transverse to the direction of wave propagation along x axis. The wave motion included helical periodic waves with continuous rotations as well as solitary waves, i.e. solitons. Another form of the equation that does not have direct physical relevance, but

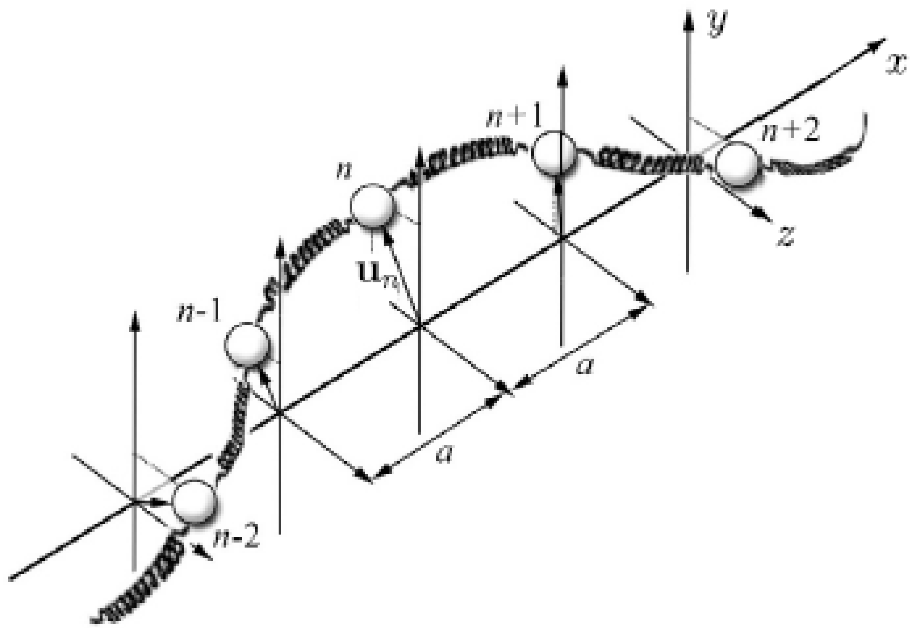


Figure 5.1: Chain of particles linked by elastic springs from [11]

.

whose solutions can be considered similar to (5.1) under certain perturbations, is given by [11, 22]:

$$\mathbf{u}_t + |\mathbf{u}|^2\mathbf{u}_x + \mathbf{u}_{xxx} = 0 \tag{5.2}$$

The above equation is considered to be a generalization of the complex modified KdV (mKdV) equation that has a Lax pair with the same scaling symmetry ($t \to \lambda^2 t, x \to \lambda x, \mathbf{u} \to \lambda^{-1}\mathbf{u}$) and hence is integrable and therefore exact analytical solution can be found [13]. In contrast for the non integrable vmKdV equation (5.1) an exact solution cannot be found, however helical solitons were constructed using dynamical systems method [11]. By appropriate choice of a parameter γ, equations (5.1) and (5.2) can be written in the following general form:

$$\mathbf{u}_t + \gamma|\mathbf{u}|^2\mathbf{u}_x + (1-\gamma)\left(|\mathbf{u}|^2\mathbf{u}\right)_x + \mathbf{u}_{xxx} = 0. \tag{5.3}$$

When $\gamma = 0$ the above equation represents non-integral vector mKdV equation (5.1) and for $\gamma = 1$ the equation represents integrable vector mKdV equation (5.2). Equation (5.1) is considered to be closely related to modified KdV equation for some positive constant k given by:

$$u_t + k|u|^2u_x + u_{xxx} = 0 \tag{5.4}$$

In order to observe the relation between (5.1) and (5.4), the equation (5.1) can be written in the following two forms [11, 23].

$$\mathbf{u}_t + |\mathbf{u}|^2\mathbf{u}_x + \mathbf{u}_{xxx} = -\mathbf{u}\left(|\mathbf{u}|^2\right)_x \tag{5.5}$$

$$\mathbf{u}_t + 3|\mathbf{u}|^2\mathbf{u}_x + \mathbf{u}_{xxx} = 2i\mathbf{u}|\mathbf{u}|^2\Theta_x \tag{5.6}$$

For the two component vector $\mathbf{u}(x,t) = r(x,t)e^{i\Theta(x,t)}$ we have $|\mathbf{u}|^2 = r^2$ and $(|\mathbf{u}|^2)_x = 2rr_x$, in two limiting cases, the right-hand side in above equations can be neglected and they reduce to integrable vmKdV equation (5.2):

1. For spatial variation of wave modulus $r = |\mathbf{u}|$ greater than spatial variation of wave argument $\Theta = \arg(\mathbf{u})$, i.e., $|r_x| \ll |r\Theta_x|$.

2. For spatial variation of wave argument $\Theta = \arg(\mathbf{u})$ greater than spatial variation of wave modulus $r = |\mathbf{u}|$, i.e., $|r\Theta_x| \ll |r_x|$.

The integrable vector mKdV equation (5.2) has an exact solution which is given by the formula [23, 11]:

$$\mathbf{u}(x,t) = \lambda\sqrt{6}\,\operatorname{sech}(\lambda\xi)\left[\cos(k\xi - \omega t + \theta_0),\ \sin(k\xi - \omega t + \theta_0)\right] \tag{5.7}$$

where $\xi = x - Vt$, k, and θ_0 are arbitrary constants, and V is the soliton speed. Another form of exact solution to Eq. (5.2) in terms of hyperbolic secant function is given by the following expression [22] (see Appendix A):

$$\mathbf{u}(x,t) = A\,\mathrm{sech}\left(\frac{x - Vt}{\Delta}\right)\left[\cos\Theta(x,t),\ \sin\Theta(x,t)\right]. \tag{5.8}$$

The parameters in equation (5.8) are defined as follows:

- Amplitude A and width $\Delta = \dfrac{\sqrt{6}}{A}$
- Velocity $V = \dfrac{A^2}{6} - 3k^2$
- Angular frequency $\omega = 2k\left(\dfrac{A^2}{6} + 4k^2\right)$, where k is an arbitrary constant
- Phase $\Theta(x,t) = k(x - Vt) - \omega t$

From the expression for the velocity V we can generate a stationary soliton by choosing, for example, $A = k\sqrt{18}$. A time evolution of one helical soliton as per Eq. (5.7) shows that the soliton propagates with a constant speed for both positive and negative values of V shown in Figure 5.2.

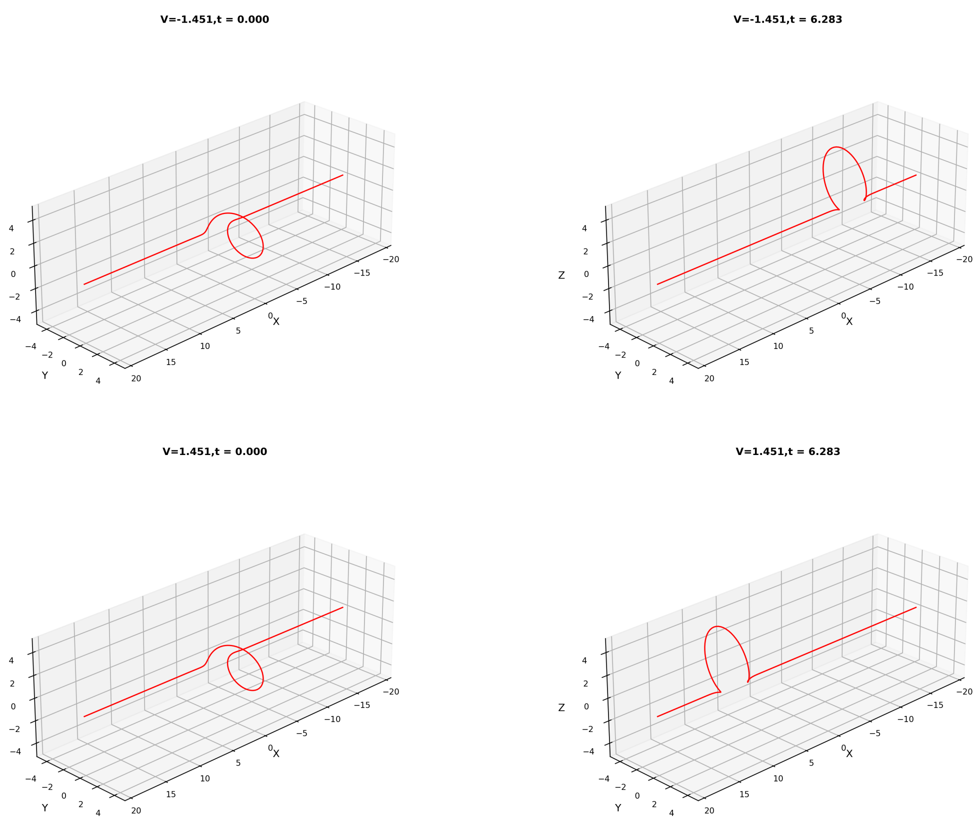


Figure 5.2: Helical soliton for $V = \pm 1.451$ ($\lambda = 2, k = 1.348, \omega = 15.683$, and $\theta_0 = 0$).

For greater helicity, i.e., the number of turns around "x" axis, the helical solitons are similar to envelope solitons for the NLS equation (4.4). The speeds of the envelope and carrier wave will be different shown in Figure 5.3.

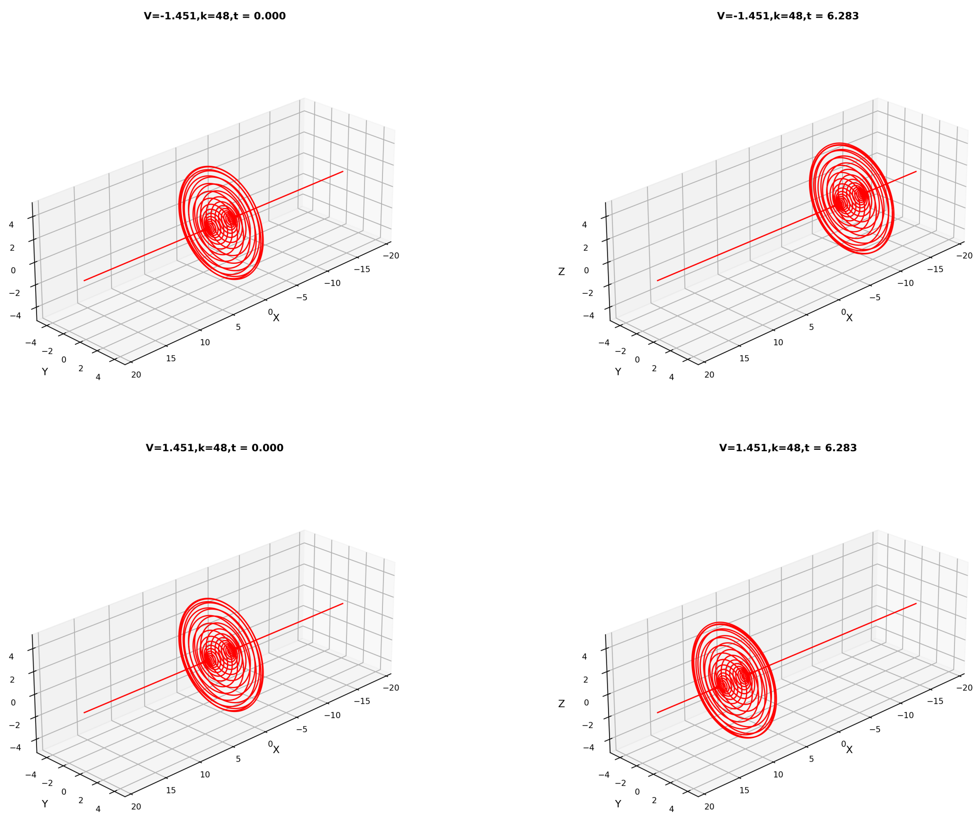


Figure 5.3: Helical soliton for $V = \pm 1.451$, $k = 48$

Both the integrable vector mKdV equation (5.2) and non-integrable vector mKdV (5.1) were solved numerically (see Appendix B for the FORTRAN code implementation) using a finite difference method (FDM). The stability of FDM numerical scheme has been validated by comparison of numerical solutions with exact analytical solutions in [31]. In addition there are alternative numerical methods developed by other authors for the solution of the vector mKdV equation, including mesh-free collocation method by Marjan et al. [28] and split-step fourier method by Erbay [29]. The numerical results generated by FORTRAN code were validated by comparison with the exact analytical solution (5.7). For the data generated by FORTRAN code (Appendix B) the following parameters were used:

Parameter	**Value**
Amplitude(A_1)	$\sqrt{6}$
k_1 (helicity)	3
Amplitude(A_2)	$0.5\sqrt{6}$
k_2 (helicity)	1

Table 5.1: Parameter values.

The interaction of helical and plane solitons at different polarization angles was discussed in [12]. The interactions of helical solitons for different helicities, both for integrable and non-integrable cases, are detailed in the next section. The system is modelled with a periodic boundary condition along a ring of a circumference L, $(0 \leq x \leq L)$. In this model, solitons can propagate in both directions along the x-axis. We consider the interaction of two solitons, the bigger soliton was set initially $x = L/4$, and the smaller soliton was set up at $x = 3L/4$. The larger soliton goes around the circumference of the ring, interacts with the smaller soliton, and propagates within the loop. For the data generated by FORTRAN code (see Appendix B), the spatial domain considered is $0 \leq x \leq 250$. This scenario of soliton propagation is common to the four types of interactions described below. Soliton interactions were simulated and can be viewed via the HTML file in a browser.

5.1.1 Interaction of helical solitons in the integrable vmKdV with the same helicity

In this type of interaction, a soliton with a smaller amplitude moves relatively faster compared to the larger soliton. Once the collision occurs, the smaller soliton is positioned behind the larger soliton and appears stationary while oscillating closer to the larger soliton shown in Figure 5.4.

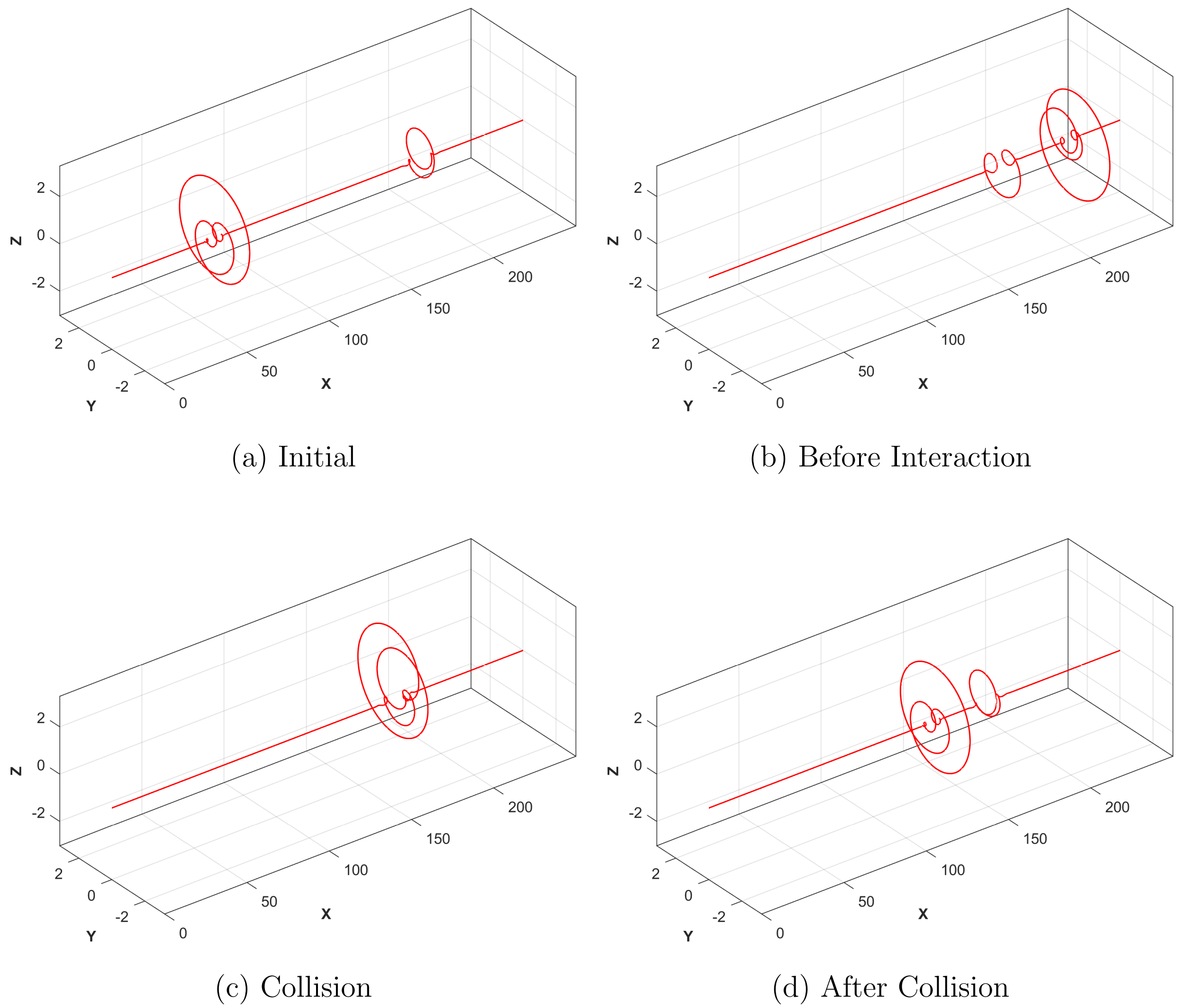
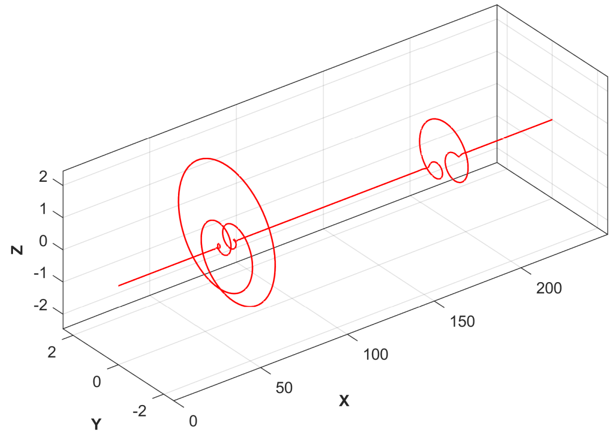


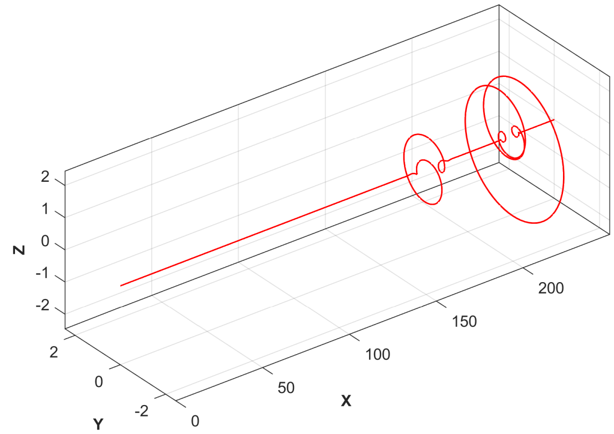
Figure 5.4: Integrable case – interaction of two helical solitons with the same helicity.

5.1.2 Interaction of helical solitons in the integrable vmKdV with the opposite helicity

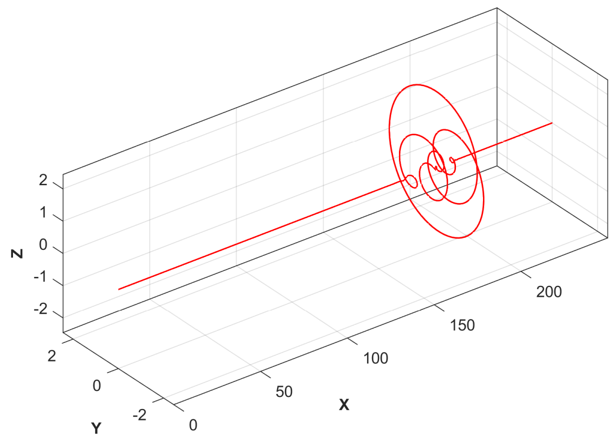
In this case, the elastic type of interaction occurred again like in the previous case. The solitons moved away after the interaction and continued moving along the circumeference of the ring. This is noticeable as the soliton with the larger amplitude moves to the left faster, leaving the smaller soliton behind. Once the collision concludes, the initial form and oscillations of the carrier wave remain the same as they were initially shown in Figure 5.5.



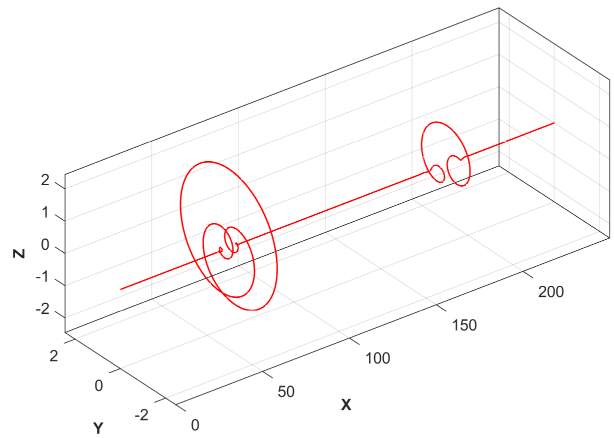
(a) Initial



(b) Before Interaction



(c) Collision

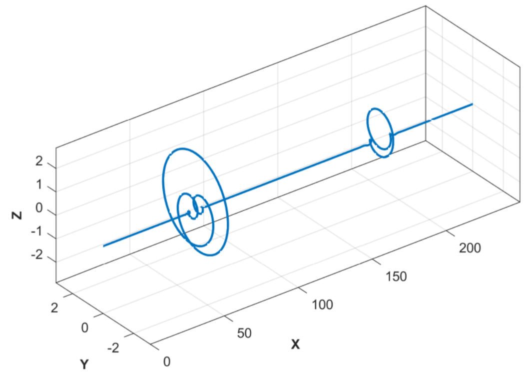


(d) After Collision

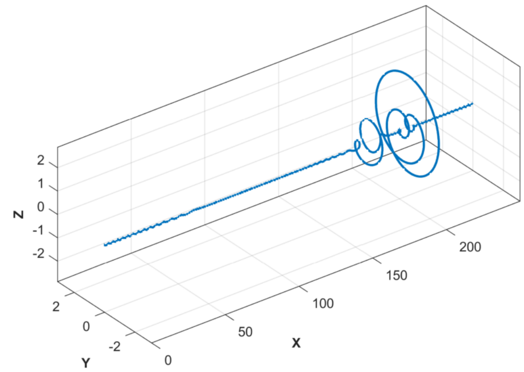
Figure 5.5: Integrable case – interaction of two helical solitons with the opposite helicity.

5.1.3 Interaction of two helical solitons in the non-integrable vmKdV equation with the same helicity

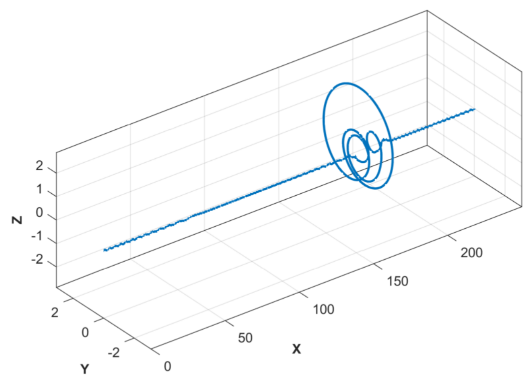
The interaction of helical solitons in the non-integrable case with the same helicity is inelastic. After the interaction, some portion of the energy transforms into a residual small amplitude wave. This is purely transient, and once the interaction is completed, the solitons propagate, preserving the original shapes shown in Figure 5.6.



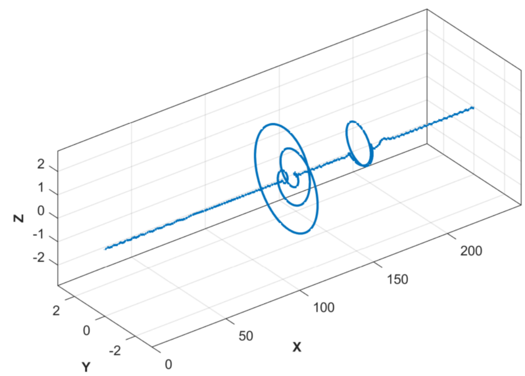
(a) Initial



(b) Before Interaction



(c) Collision

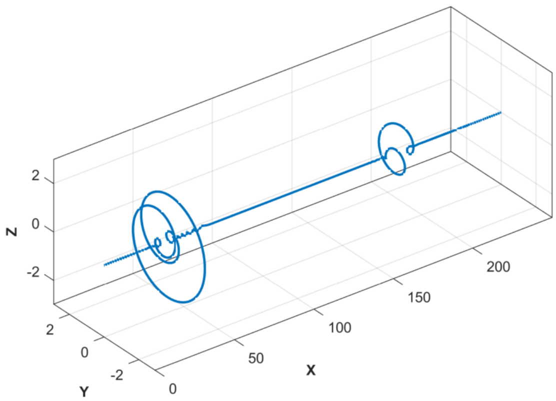


(d) After Collision

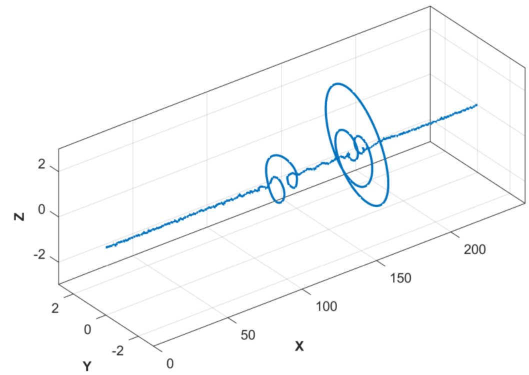
Figure 5.6: Non-integrable case – interaction of two helical solitons with the same helicity.

5.1.4 Interaction of two helical solitons in the non-integrable vmKdV equation with the opposite helicity

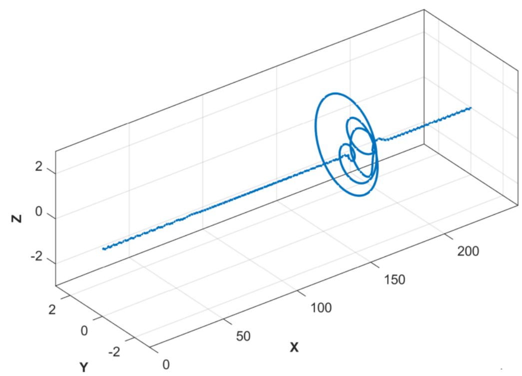
In this type of interaction, there are disturbances or ripples along the x-axis. When solitons collide, some portion of energy is radiated, and small ripples appear behind solitons. This indicates that the interaction is inelastic. Thereafter, the solitons move away from each other, retaining their shapes but not amplitudes shown in Figure 5.7.



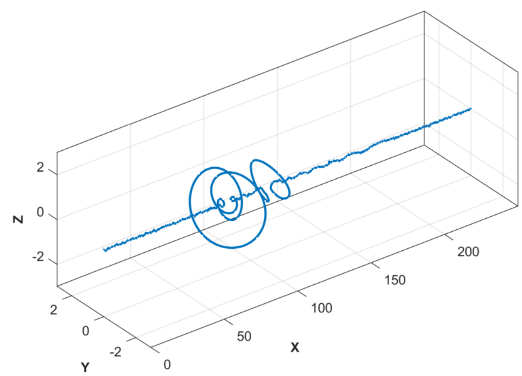
(a) Initial



(b) Before Interaction



(c) Collision



(d) After Collision

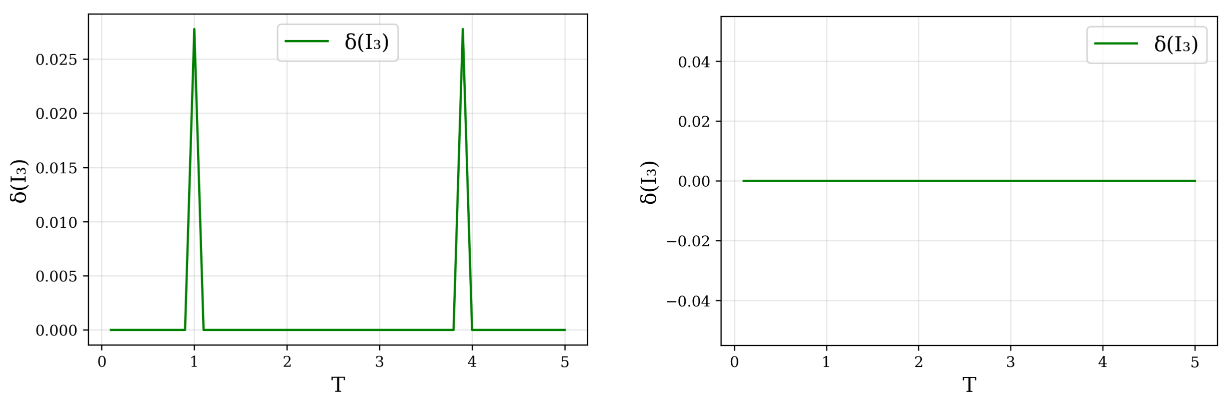
Figure 5.7: Non-integrable case – interaction of two helical solitons with the opposite helicity.

5.2 Numerical results and discussion

Numerical calculations confirm the stability and robustness of helical solitons both in the integrable and non integrable cases of the vector mKdV equations (5.2) and (5.1) using the same initial conditions. The energy density are evaluated to ensure the stability of numerical scheme. The energy density conservation is determined as $I_3 = \int |\mathbf{u}|^2 dx$, in the domain $0 \leq x \leq 250$, and it is noted that energy density is preserved in numerical calculations with a relative error of less than 0.03%. The helical soliton in the integrable case propagates with constant speed; this agrees with theoretical predictions and holds true for solitons propagating with the same or opposite helicity. In order to examine the temporal stability of I_3, the following formula was used to calculate the relative error:

$$\delta(I_3) = \frac{|I_3(t) - I_3(0)|}{I_3(0)} \times 100\,\% \tag{5.9}$$

In the integrable case, for helical soliton interactions with the same and opposite helicities there was an almost identical conservation of I_3 with zero relative error at most data points $0 \leq T \leq 5$ expect in the case of the same helicity at two instants of time at $T \approx 1$ and $T \approx 3.8$ when the percentage of relative error attains $\delta(I_3) \approx 0.027\%$; this happens during the process of soliton interaction. The result indicates that, in the integrable vector mKdV equation, energy is conserved and is shown in Figure 5.8.

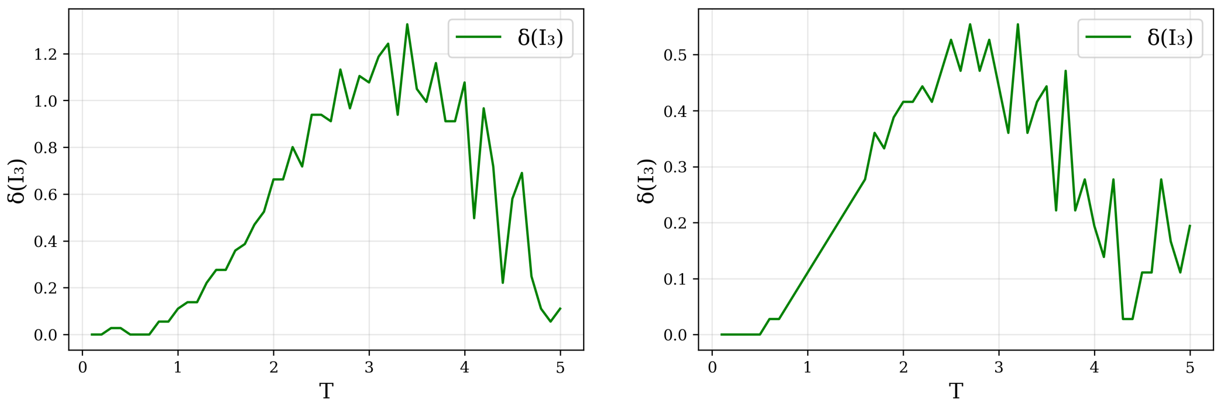


(a) Same helicity

(b) Opposite helicity

Figure 5.8: Integrable case – relative error $\delta(I_3)\%$ vs. time T

For the same initial condition in the non integrable case with the same helicity, there are small disturbances which appear behind each soliton in the course of propagation; they are noticeable at $T \approx 0.3$ with $\delta(I_3) \approx 0.0276$. Thereafter, the variation is less pronounced shown in Figure 5.9(a). In contrast, for the non integrable case with opposite helicity, the relative error $\delta(I_3)$ is very close to zero up to $T \approx 1.7$ when the variation of $\delta(I_3)$ becomes visible. This is caused by the energy radiation before and after the interaction shown in Figure 5.9(b).



(a) Same helicity (b) Opposite helicity

Figure 5.9: Non integrable case – relative error $\delta(I_3)\%$ vs. time T

The choice of the parameter k in Eq. (5.8) in the initial condition establishes the presence of helical solitons in integrable and non integrable case.s For the same initial conditions with $k \approx 1.348, 1.155$ the integrable case presents helical solitons and agrees with the theoretical prediction. In contrast to that, in the non-integrable case, the helical solitons disintegrates into several plane solitons propagating at different angles [11] and is shown in Figure 5.10.

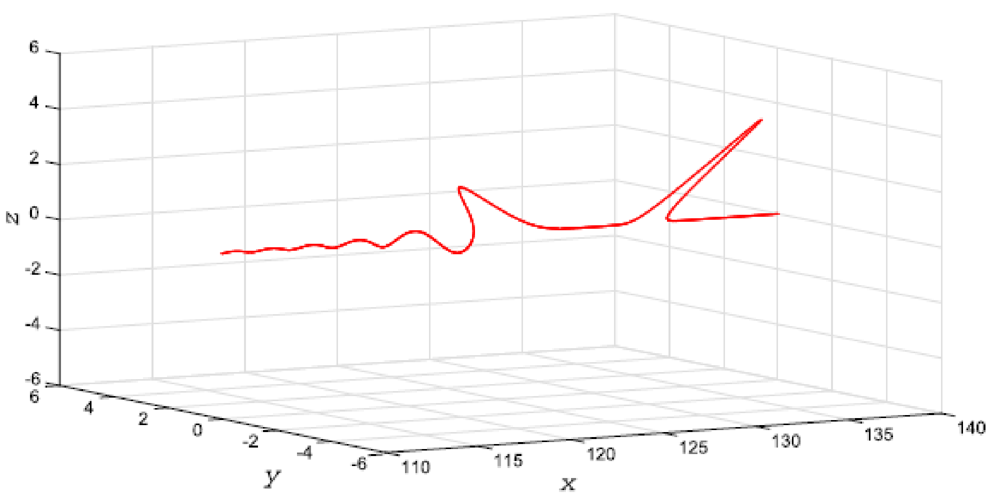


Figure 5.10: Non integrable case plane solitons (From [11]).

However, when the parameter k becomes large, $k \gg 1$, the difference between the integrable and non-integrable cases becomes insignificant; therefore interaction of helical solitons with large k

should be almost identical. This issue requires further investigation.

Chapter 6

Conclusion

In this literature review, helical solitons were considered for two different forms of the vector modified KdV equation one of which is integrable and another is non-integrable. The integrable vmKdV equation has an exact analytical solution and numerical simulation confirmed the presence of helical solitons that matches theoretical prediction. However the integrable vmKdV equation did not find application in physical sciences thus far. The non integrable form of the vmKdV equation was considered for a modelling of various physical systems, for example, chain of particles connected by a string. Numerical simulations confirmed the presence of helical solitons only for negative velocities. In contrast to that, in the integrable vmKdV equation helical solitons can propagate both with positive and negative velocities. The interaction of helical solitons in both forms of equation were considered for the same and opposite helicities. The numerical simulations confirm that soliton interactions are elastic in the integrable vmKdV equation but is inelastic in the non-integrable vmKdV equation. A further area of research to be considered is the interaction of helical solitions in a large domain with bigger helicities.

Appendix A

Integrable vmKdV solution

This appendix derives the analytical solution for the integrable vector mKdV equation referenced in Chapter 5. The integrable vector mKdV equation is given by:

$$\mathbf{u}_t + |\mathbf{u}|^2\mathbf{u}_x + \mathbf{u}_{xxx} = 0, \quad \text{where} \quad \mathbf{u} = u_1 + iu_2 = r(x,t)[\cos\Theta(x,t), \sin\Theta(x,t)] \tag{A.1}$$

Assume a travelling wave solution for Eq. (A.1) of the form:

$$\mathbf{u}(x,t) = \mathbf{U}(x - Vt) = f(\xi)[\cos\Theta(x,t), \sin\Theta(x,t)] \tag{A.2}$$

where $\xi = x - Vt, \Theta(x,t) = k(x - Vt) - \omega t$. The partial derivatives are listed in the table below:

Function	Partial Derivatives	Value	Comments
ξ	$\partial_x(\xi)$, $\partial_t(\xi)$	1, $-V$	
Θ	$\partial_x(\Theta)$, $\partial_t(\Theta)$	k, $-kV - \omega$	
$f(\xi)$	$\partial_t(f(\xi))$, $\partial_x(f(\xi))$	$-Vf'$, f'	$f' = \frac{df}{d\xi}$
$f(\xi)$	$\partial_{xx}(f(\xi))$, $\partial_{xxx}(f(\xi))$	f'', f'''	$f'' = \frac{d^2f}{d\xi^2}$, $f''' = \frac{d^3f}{d\xi^3}$

The first partial derivative of $\mathbf{u}$ with respect to t is

$$\mathbf{u}_t = \partial_t\big(f(\cos\Theta, \sin\Theta)\big) \tag{A.3}$$

$$= -Vf'(\cos\Theta, \sin\Theta) + f\Theta_t(-\sin\Theta, \cos\Theta), \tag{A.4}$$

Substituting for Θ_t, we obtain:

$$\mathbf{u}_t = -Vf'(\cos\Theta, \sin\Theta) + f(-kV - \omega)(-\sin\Theta, \cos\Theta). \tag{A.5}$$

The first partial derivative of $\mathbf{u}$ with respect to x is

$$\mathbf{u}_x = \partial_x\big(f(\cos\Theta, \sin\Theta)\big) \tag{A.6}$$

$$= f'(\cos\Theta, \sin\Theta) + f\Theta_x(-\sin\Theta, \cos\Theta). \tag{A.7}$$

Substituting for Θ_x, we obtain:

$$\mathbf{u}_x = f'(\cos\Theta, \sin\Theta) + fk(-\sin\Theta, \cos\Theta), \tag{A.8}$$

The second partial derivative of $\mathbf{u}$ with respect to x is

$$\mathbf{u}_{xx} = \partial_x\big(f'(\cos\Theta, \sin\Theta) + fk(-\sin\Theta, \cos\Theta)\big) \tag{A.9}$$

$$= f''(\cos\Theta, \sin\Theta) + f'\Theta_x(-\sin\Theta, \cos\Theta) + f'k(-\sin\Theta, \cos\Theta) + f\Theta_x(-cos\Theta, -\sin\Theta). \tag{A.10}$$

Substituting for Θ_x, we obtain:

$$\mathbf{u}_{xx} = f''(\cos\Theta, \sin\Theta) + 2kf'(-\sin\Theta, \cos\Theta) + k^2 f(-\cos\Theta, -\sin\Theta). \tag{A.11}$$

The third partial derivative of $\mathbf{u}$ with respect to x is

$$\mathbf{u}_{xxx} = \partial_x\big(f''(\cos\Theta, \sin\Theta) + 2kf'(-\sin\Theta, \cos\Theta) + k^2 f(-\cos\Theta, -\sin\Theta)\big) \tag{A.12}$$

$$= f'''(\cos\Theta, \sin\Theta) + f''\Theta_x(-\sin\Theta, \cos\Theta) + 2kf''(-\sin\Theta, \cos\Theta) \tag{A.13}$$

$$+ 2kf'\Theta_x(-\cos\Theta, -\sin\Theta) + k^2 f'(-\cos\Theta, -\sin\Theta) + k^2 f\Theta_x(\sin\Theta, -\cos\Theta). \tag{A.14}$$

Substituting for $\Theta_x = k$, we obtain:

$$\mathbf{u}_{xxx} = f'''(\cos\Theta, \sin\Theta) + 3kf''(-\sin\Theta, \cos\Theta) - \underbrace{3k^2 f'(\cos\Theta, \sin\Theta)}_{\text{factoring out } -1 \text{ for } (\cos\Theta, \sin\Theta)} . \tag{A.15}$$

The terms $(\cos\Theta, \sin\Theta)$ and $(-\sin\Theta, \cos\Theta)$ are orthogonal since their dot product is zero. The partial derivatives $\mathbf{u}_t, \mathbf{u}_x, \mathbf{u}_{xxx}$ are represented in simpler form by considering $\mathbf{v} = (\cos\Theta, \sin\Theta)$, and $\mathbf{v}_\perp = (-\sin\Theta, \cos\Theta)$; therefore, Eqs. (A.8), (A.5), and (A.15) simplify to:

$$\mathbf{u}_t = -Vf'\mathbf{v} + (-kV - \omega)f\mathbf{v}_\perp, \tag{A.16}$$

$$\mathbf{u}_x = f'\mathbf{v} + kf\mathbf{v}_\perp, \tag{A.17}$$

$$\mathbf{u}_{xxx} = (f''' - 3k^2 f')\mathbf{v} + 3kf''\mathbf{v}_\perp. \tag{A.18}$$

Using $|\mathbf{u}|^2 = f^2$, Eq. (A.1) becomes:

$$-Vf'\mathbf{v} + f^2 f'\mathbf{v} + f^3 k\mathbf{v}_\perp + f'''\mathbf{v} + 3kf''\mathbf{v}_\perp - 3k^2 f'\mathbf{v} = 0. \tag{A.19}$$

Grouping the terms for $\mathbf{v}$, $\mathbf{v}_\perp$ and after rearranging , we obtain:

$$(f''' - (V + 3k^2)f' + f^2 f')\mathbf{v} + (f^3 k + 3kf'')\mathbf{v}_\perp = 0. \tag{A.20}$$

This results in two equations

$$(f''' - (V + 3k^2)f' + f^2 f') = 0, \tag{A.21}$$

$$f^3 k + 3kf'' = 0. \tag{A.22}$$

Consider Eq. (A.21); multiplying it by f' and noting that $f' = \frac{df}{d\xi}$ we obtain:

$$f'f'' - (V + 3k^2)ff' + \frac{f^3}{3}f' = 0, \tag{A.23}$$

$$\frac{d}{d\xi}\left(\frac{f'^2}{2}\right) - (v + 3k^2)\frac{d}{d\xi}\left(\frac{f^2}{2}\right) + \frac{d}{d\xi}\left(\frac{f^4}{12}\right) = 0. \tag{A.24}$$

Integrating with respect to ξ, we obtain:

$$\frac{(f')^2}{2} - (V + 3k^2)\frac{f^2}{2} + \frac{f^4}{12} = K, \quad \text{where} \quad K \text{ is constant.} \tag{A.25}$$

For a localized solution such that $f \to 0, f' \to 0,$ when $|\xi| \to \infty$ and hence $K = 0$, the above equation can be simplified to:

$$(f')^2 = (V + 3k^2)f^2 - \frac{f^4}{6} = f^2 \left[(V + 3k^2) - \frac{f^2}{6}\right], \quad \text{or finally} \tag{A.26}$$

$$\frac{df}{d\xi} = \pm f\sqrt{(V + 3k^2) - \frac{f^2}{6}} \tag{A.27}$$

Equation (A.27) can be solved separation of variables and integrating both sides:

$$\int \frac{df}{f\sqrt{(V + 3k^2) - \frac{f^2}{6}}} = \pm \int d\xi = \pm\xi + C. \tag{A.28}$$

The integral equation (A.28) can be analytically evaluated and has the well-know solution with the sech profile:

$$f(\xi) = A\,\text{sech}\left(\frac{\xi}{\Delta}\right), \tag{A.29}$$

where the amplitude $A = \sqrt{6(V + 3k^2)}$ and half-width $\Delta = \sqrt{6}/A$. The final expression for the one-soliton solution of the integrable vector mKdV Eq. (A.1) is:

$$\mathbf{u}(x,t) = A\,\text{sech}\left(\frac{x - Vt}{\Delta}\right)\big[\cos\Theta(x,t), \sin\Theta(x,t)\big]. \tag{A.30}$$

This solution with different parameters will be used for the creation of initial conditions for integrable and non-integrable versions of vector mKdV equations.

Appendix B

Program VMKDV

This appendix contains the complete Fortran source code for the VMKDV (Vector Modified Korteweg–de Vries) program used in the numerical simulations.

```
C********!*********!*********!*********!*********!*********!*********!**
C Spatial step H = RL/N, temporal step TAU
C Step for data presentation: DT = NB*TAU
C NWAY = 1: for analitically given initial conditions
C NWAY = 2: for numerical initial conditions
C Criterion of stability of the numerical scheme: TAU <, = 0.384*H**3/B
C Used markers: 1-21
C********!*********!*********!*********!*********!*********!*********!**
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION P(100004),U(100004),V(100000),S(100001),
     *P1(100004),U1(100004),V1(100000),S1(100001),X(100000)
      EQUIVALENCE (V(1),U(3)),(V1(1),U1(3))
      COMMON/SLT/A,A1,B,RL
      OPEN(10,FILE='Work1.DAT')
      OPEN(11,FILE='Work2.DAT')
      OPEN(12,FILE='Work3.DAT')
      OPEN(13,FILE='Work4.DAT')
      OPEN(14,FILE='Work5.DAT')
      OPEN(15,FILE='Work6.DAT')
      OPEN(16,FILE='Work7.DAT')
```

```
      OPEN(17,FILE='Work8.DAT')
      OPEN(18,FILE='Work9.DAT')
      OPEN(19,FILE='Work10.DAT')
      OPEN(20,FILE='Work11.DAT')
      OPEN(21,FILE='Work12.DAT')
      OPEN(22,FILE='Work13.DAT')
      OPEN(23,FILE='Work14.DAT')
      OPEN(24,FILE='Work15.DAT')
      OPEN(25,FILE='Work16.DAT')
      OPEN(26,FILE='Work17.DAT')
      OPEN(27,FILE='Work18.DAT')
      OPEN(28,FILE='Work19.DAT')
      OPEN(29,FILE='Work20.DAT')
      OPEN(30,FILE='Work21.DAT')
      OPEN(32,FILE='Work22.DAT')
      OPEN(33,FILE='Work23.DAT')
      OPEN(34,FILE='Work24.DAT')
      OPEN(35,FILE='Work25.DAT')
      OPEN(36,FILE='Work26.DAT')
      OPEN(37,FILE='Work27.DAT')
      OPEN(38,FILE='Work28.DAT')
      OPEN(39,FILE='Work29.DAT')
      OPEN(40,FILE='Work30.DAT')
      OPEN(41,FILE='Work31.DAT')
      OPEN(42,FILE='Work32.DAT')
      OPEN(43,FILE='Work33.DAT')
      OPEN(44,FILE='Work34.DAT')
      OPEN(45,FILE='Work35.DAT')
      OPEN(46,FILE='Work36.DAT')
      OPEN(47,FILE='Work37.DAT')
      OPEN(48,FILE='Work38.DAT')
      OPEN(49,FILE='Work39.DAT')
      OPEN(50,FILE='Work40.DAT')
      OPEN(51,FILE='Work41.DAT')
      OPEN(52,FILE='Work42.DAT')
      OPEN(53,FILE='Work43.DAT')
```

```
      OPEN(54,FILE='Work44.DAT')
      OPEN(55,FILE='Work45.DAT')
      OPEN(56,FILE='Work46.DAT')
      OPEN(57,FILE='Work47.DAT')
      OPEN(58,FILE='Work48.DAT')
      OPEN(59,FILE='Work49.DAT')
      OPEN(60,FILE='Work50.DAT')
      OPEN(101,FILE='CONTIN.DAT') ! DATA FILE FOR RUN CONTINUATION WHEN NWAY=2
      OPEN(102,FILE='INTEGRALS.DAT') ! INTEGRAL QUANTITIES
      OPEN(103,FILE='MINIMAX.DAT') ! MAXIMA AND MINIMA
      OPEN(104,FILE='NEXTCALCUL.DAT') ! DATA STORED FOR THE NEXT RUN
C********!*********!*********!*********!*********!*********!*********!**
C Parameters
      A=1.D0
      A1=1!1!1.D0 ! For the non-integrable case A1=1 whereas for the integrable
    case A1=0
      B=1.D0
      RL=2.5D2!5.0D0
      N=10000!200
      H=RL/N
      TAUcr=3.84D-1*H**3/B
      TAU=5.0D-6!1.0D-6!5.0D-6
      DT=1.0D-1!5.0D-1!5.0D-4
      NP=50
         PRINT 8
  8      FORMAT(/1X,'NWAY = ?'/
     *   '   1 - FOR ANALYTICALY GIVEN INITIAL CONDITION;'/
     *   '   2 - FOR NUMERICALY GIVEN INITIAL CONDITION   ')
      READ(*,*) NWAY
      write(*,*) 'A = ',A, 'A1 = ',A1
      write(*,*) 'B = ',B, 'Rl = ',RL
      write(*,*) 'TAU = ',TAU, 'TAUcr = ',TAUcr
      write(*,*) 'H = ',H, 'DT = ',DT
      write(*,*) 'N = ',N, '     NP = ',NP
      PAUSE
C********!*********!*********!*********!*********!*********!*********!**
```

```
      NB=DT/TAU+0.00001
      write(*,*) 'NB = ',NB, '     NWAY = ',NWAY
      N1=N+1
      N2=N+2
      N3=N+3
      N4=N+4
      DO 11 IR=1,N
 11   X(IR)=RL*(IR-1)/N
C*******!*********!*********!*********!*********!*********!*********!**
C Initial conditions and first step
      GO TO (1,2), NWAY
  1   CALL STEP1(N,U,P,U1,P1,S,S1,A,A1,B,TAU,H,N1,N2,N3,N4)
      T=0.D0
      M=0
      NB1=NB-1
C*******!*********!*********!*********!*********!*********!*********!**
C Momentum and energy
      DO 34 L=1,N
      S(L)=P(L)**2
      S1(L)=P1(L)**2
 34   CONTINUE
      SI1=0.D0
      SI2=0.D0
      SE1=0.D0
      SE2=0.D0
      DO 81 IJ=1,N
      SE1=SE1+S(IJ)
      SI1=SI1+P(IJ)
      SI2=SI2+P1(IJ)
      SE2=SE2+S1(IJ)
 81   CONTINUE
      QI1=SI1/N
      QI2=SI2/N
      QES=5.D-1*(SE1+SE2)/N
      UMA=P(1)
      UMA1=P1(1)
```

```
      UMI=P(1)
      UMI1=P1(1)
      DO 334 I=2,N
      IF (P(I).LE.UMA) GO TO 333
      UMA=P(I)
      LVMAX=I
 333  IF(P(I).GE.UMI) GO TO 334
      UMI=P(I)
      LVMIN=I
 334  CONTINUE
      DO 335 IM=2,N
      IF(P1(IM).LE.UMA1) GO TO 336
      UMA1=P1(IM)
      LU1MA=IM
 336  IF (P1(IM).GE.UMI1) GO TO 335
      UMI1=P1(IM)
      LU1MI=IM
 335  CONTINUE
      XMA=(LVMAX-1)*H
      XMI=(LVMIN-1)*H
      X1MA=(LU1MA-1)*H
      X1MI=(LU1MI-1)*H
      PRINT 9,T,M,QI1,QI2,QES,UMA,XMA,UMI,XMI,UMA1,X1MA,UMI1,X1MI
      DO 10 J=1,N
      S(J)=DSQRT(P(J+2)**2+P1(J+2)**2)
  10  CONTINUE
C      DO 15 J=1,N
C      S1(J)=0.
C      IF(H*J.LT.35.) GO TO 15
C      S1(J)=ATAN2(P1(J+2),P(J+2))
C      IF(H*J.GT.130.) S1(J)=PI/2.
C  15  CONTINUE
      DO 337 IX=1,N
      WRITE(10,*) X(IX),P(IX+2),P1(IX+2),S(IX)
 337  CONTINUE
      GO TO 314
```

```
C********!*********!*********!*********!*********!*********!*********!**
    2   T = 0.D0
        DO 300 LN=1,N4
        READ(101,*) P(LN),U(LN),P1(LN),U1(LN)
  300   CONTINUE
C        PAUSE
        NB1=NB
        PRINT 9,T
C        PAUSE
C********!*********!*********!*********!*********!*********!*********!**
  314   CONTINUE
        WRITE(102,9) T,M,QI1,QI2,QES,UMA,XMA,UMI,XMI,UMA1,X1MA,UMI1,X1MI
        DO 4 I=1,NP
        WRITE(*,*) 'NSTEP = ',I, 'T = ', T
        DO 5 IA=1,NB1
        UB1=U(1)
        UV1=U1(1)
        YU=U(2)
        YU1=U1(2)
        UF1=U(3)
        UFE1=U1(3)
        UF2=U(4)
        UFE2=U1(4)
        DO 6 J=3,N2
        UB2=UB1
        UB1=YU
        YU=UF1
        UF1=UF2
        UF2=U(J+2)
        UV2=UV1
        UV1=YU1
        YU1=UFE1
        UFE1=UFE2
        UFE2=U1(J+2)
C********!*********!*********!*********!*********!*********!*********!**
        P(J)=P(J)-(TAU/H)*((B/H**2)*(UF2-2.D0*UF1+2.D0*UB1-UB2)
```

```
     &+A*(YU**2+YU1**2+2.D0*A1*A*YU**2)*(UF1-UB1)+2.D0*A1*A*YU1*YU*
     &(UFE1-UV1))
      P1(J)=P1(J)-(TAU/H)*((B/H**2)*(UFE2-2.D0*UFE1+2.D0*UV1-UV2)
     &+A*(YU1**2+YU**2+2.D0*A1*A*YU1**2)*(UFE1-UV1)+2.D0*A1*A*YU1*YU*
     &(UF1-UB1))
  6   CONTINUE
      DO 7 K=3,N2
      W=P(K)
      P(K)=U(K)
      W1=P1(K)
      P1(K)=U1(K)
      U1(K)=W1
   7  U(K)=W
      U(1)=U(N1)
      U(2)=U(N2)
      U(N3)=U(3)
      U(N4)=U(4)
      U1(1)=U1(N1)
      U1(2)=U1(N2)
      U1(N3)=U1(3)
      U1(N4)=U1(4)
   5  CONTINUE
      NB1=NB
C*******!*********!*********!*********!*********!*********!*********!**
      DO 234 L=1,N
      S(L)=U(L+2)**2
      S1(L)=U1(L+2)**2
 234  CONTINUE
      SI1=0.
      SI2=0.
      SE1=0.
      SE2=0.
      DO 281 IJ=1,N
      SE1=SE1+S(IJ)
      SI1=SI1+V(IJ)
      SI2=SI2+V1(IJ)
```

```
      SE2=SE2+S1(IJ)
 281  CONTINUE
      QI1=SI1/N
      QI2=SI2/N
      QES=5.D-1*(SE1+SE2)/N
      UMA=V(1)
      UMA1=V1(1)
      UMI=V(1)
      UMI1=V1(1)
      DO 134 I0=2,N
      IF (V(I0).LE.UMA) GO TO 133
      UMA=V(I0)
      LVMAX=I0
 133  IF(V(I0).GE.UMI) GO TO 134
      UMI=V(I0)
      LVMIN=I0
 134  CONTINUE
      DO 135 IM=2,N
      IF (V1(IM).LE.UMA1) GO TO 136
      UMA1=V1(IM)
      LU1MA=IM
 136  IF(V1(IM).GE.UMI1) GO TO 135
      UMI1=V1(IM)
      LU1MI=IM
 135  CONTINUE
      XMA=(LVMAX-1)*H
      XMI=(LVMIN-1)*H
      X1MA=(LU1MA-1)*H
      X1MI=(LU1MI-1)*H
      M=M+NB
      T=T+DT
C      PRINT 9,T,M,QI1,QI2,QES,UMA,XMA,UMI,XMI,UMA1,X1MA,UMI1,X1MI
      WRITE(102,22) T,QI1,QI2,QES
      WRITE(103,23) T,UMA,XMA,UMI,XMI,UMA1,X1MA,UMI1,X1MI
      DO 110 J=1,N
      S(J)=DSQRT(V(J)**2+V1(J)**2)
```

```
  110 CONTINUE
C       DO 115 J=1,N
C       S1(J)=0.
C       IF(H*J.LT.35.) GO TO 115
C       S1(J)=ATAN2(V1(J),V(J))
C       IF(H*J.GT.130.) S1(J)=PI/2.
C   115 CONTINUE
      DO 237 IX=1,N
      WRITE(10+I,*) X(IX),P(IX+2),P1(IX+2),S(IX)
 237  CONTINUE
  4   CONTINUE
      DO 200 LN=1,N4
      WRITE(104,*) P(LN),U(LN),P1(LN),U1(LN)
 200  CONTINUE
      CLOSE(104)
C*******!*********!*********!*********!*********!*********!*********!**
      STOP
   9  FORMAT(/1X,2HT=,G10.4,3X,2HM=,I7,3X,'QI1=',G10.4,3X,'QI2=',G10.4,
     &3X,'QES=',G16.10,1X,'UMA= ',G10.4,3X,'XMA= ',G10.4,'UMI= ',G10.4,
     &3X,'XMI=',G10.4,1X,'U1MA=',G10.4,3X,'X1MA=',G10.4,'U1MI=',G10.4,
     &3X,'X1MI=',G10.4)
  22  FORMAT(4E12.4)
  23  FORMAT(9E12.4)
      END
C*******!*********!*********!*********!*********!*********!*********!**
      SUBROUTINE INFUN(S,S1,N,H)
C INITIAL CONDITION
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION S(1),S1(1)
      COMMON/SLT/A,A1,B,RL
      PI = 3.14159265359D0
C       AMP =2.0D0*DSQRT(6.D0)!0.0D0
      AMP =1.0D0*DSQRT(6.D0)!0.0D0
C     AK0 = 1.348D0 !For helicity
      AK0 = 3.D0 !For helicity
C     AK0 = 0.D0
```

```
      DLT1 = DSQRT(6.D0*B/A)/AMP ! - for the helical soliton
C     DLT1 = DSQRT(2.D0*B/A)/AMP ! - for the plane nonintegrable soliton
C     AMQ = 0.D0 ! Amplitude of the second soliton
      AMQ = 5.D-1*DSQRT(6.D0) ! Amplitude of the second soliton
      DLT2 = DSQRT(6.D0*B/A)/AMQ ! Width of the second soliton
      AK2 = 1.0D0 ! Carrier wavenumber of the second soliton
C     AK2 = 0.D0 ! - for the plane soliton
C      PHI2 = PI/6.D0 ! - for the plane soliton at the angle
C      PHI1 = 0.D0
      DO 2 K=1,N
      X1 = H*K - 2.5D-1*RL
      X2 = H*K - 7.5D-1*RL
      PHI1 = AK0*X1
      PHI2 = AK2*X2 ! for the helical soliton
      S(K) = AMP*DSQRT(1.D0 - (DTANH(X1/DLT1))**2)*DCOS(PHI1)
     &+ AMQ*DSQRT(1.D0 - (DTANH(X2/DLT2))**2)*DCOS(PHI2)
      S1(K) = AMP*DSQRT(1.D0 - (DTANH(X1/DLT1))**2)*DSIN(PHI1)
     &+ AMQ*DSQRT(1.D0 - (DTANH(X2/DLT2))**2)*DSIN(PHI2)
    2 CONTINUE
      RETURN
      END

C********!*********!*********!*********!*********!*********!*********!**
      SUBROUTINE STEP1(N,U,P,U1,P1,S,S1,A,A1,B,TAU,H,N1,N2,N3,N4)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION U(1),P(1),S(1),U1(1),P1(1),S1(1)
      EPS=1.D-3
      T1=TAU
      ASSIGN 11 TO JB
      KB=1
      DO 7 L=1,N4
      U1(L)=0.D0
    7 U(L)=0.D0
C********!*********!*********!*********!*********!*********!*********!**
    9 CALL INFUN(S,S1,N,H)
      DO 10 M=1,N
```

```
      P1(M+2)=S1(M)
 10   P(M+2)=S(M)
      P(1)=P(N1)
      P(2)=P(N2)
      P(N3)=P(3)
      P(N4)=P(4)
      P1(1)=P1(N1)
      P1(2)=P1(N2)
      P1(N3)=P1(3)
      P1(N4)=P1(4)
      GO TO JB
C********!*********!*********!*********!*********!*********!*********!**
  11  T1=5.D-1*T1
      KB=KB*2
      DO 6 I=1,KB
      B1=P(1)
      PE=P(2)
      F1=P(3)
      F2=P(4)
      V1=P1(1)
      PE1=P1(2)
      FE1=P1(3)
      FE2=P1(4)
      DO 8 J=3,N2
      B2=B1
      B1=PE
      PE=F1
      F1=F2
      F2=P(J+2)
      V2=V1
      V1=PE1
      PE1=FE1
      FE1=FE2
      FE2=P1(J+2)
C********!*********!*********!*********!*********!*********!*********!**
      P1(J)=PE1-T1*(B*(FE2-2.D0*FE1+2.D0*V1-V2)+H**2*(A*(PE1**2+PE**2)+
```

```
     &2.D0*A*A1*PE1**2)*(FE1-V1)+2.D0*A1*A*H**2*PE*PE1*(F1-B1))/H**3
    8 P(J)=PE-T1*(B*(F2-2.D0*F1+2.D0*B1-B2)+H**2*(A*(PE**2+PE1**2)+
     &2.D0*A*A1*PE**2)*(F1-B1)+2.D0*A1*A*H**2*PE*PE1*(FE1-V1))/H**3
      P(1)=P(N1)
      P(2)=P(N2)
      P(N3)=P(3)
      P(N4)=P(4)
      P1(1)=P1(N1)
      P1(2)=P1(N2)
      P1(N3)=P1(3)
      P1(N4)=P1(4)
    6 CONTINUE
C********!*********!*********!*********!*********!*********!*********!**
      E=0.D0
      DO 5 JE=3,N2
      E=E+(P(JE)-U(JE))**2+(P1(JE)-U1(JE))**2
    5 CONTINUE
      E=DSQRT(E)
      PRINT 1,T1,E
      DO 4 MA=1,N4
      U1(MA)=P1(MA)
    4 U(MA)=P(MA)
      IF(E-EPS) 3,3,9
    3 ASSIGN 2 TO JB
      GO TO 9
C********!*********!*********!*********!*********!*********!*********!**
    2 CONTINUE
    1 FORMAT(5X,3HT1=,E10.4,5X,2HE=,E10.4)
      RETURN
      END
```

Listing B.1: Fortran code for VMKDV program

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